

ON Q -INTEGRAL $(3, s)$ -SEMIREGULAR BIPARTITE GRAPHS

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A graph is called Q -integral if its signless Laplacian spectrum consists entirely of integers. We establish some general results regarding signless Laplacians of semiregular bipartite graphs. Especially, we consider those semiregular bipartite graphs with integral signless Laplacian spectrum. In some particular cases we determine the possible Q -spectra and consider the corresponding graphs.

1. INTRODUCTION

Let G be a simple graph with adjacency matrix A ($= A_G$). The eigenvalues and the spectrum of A are also called the *eigenvalues* and the *spectrum* of G , respectively. If we consider a matrix $Q = D + A$ instead of A , where D is the diagonal matrix of vertex-degrees (in G), we get the *signless Laplacian eigenvalues* and the *signless Laplacian spectrum*, respectively. For short, the signless Laplacian eigenvalues and the signless Laplacian spectrum will be called the Q -*eigenvalues* and the Q -*spectrum*, respectively. We say that a graph is Q -*integral* if its signless Laplacian spectrum consists entirely of integers.

Let R ($= R_G$) be the $n \times m$ vertex-edge incidence matrix of G . Denote by $L(G)$ the *line graph* of G (recall, vertices of $L(G)$ are in one-to-one correspondence with edges of G , and two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent). The following relations are well known (see, for example, [2]):

$$RR^T = A_G + D, \quad R^T R = A_{L(G)} + 2I.$$

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From these relations it immediately follows that

$$(1) \quad P_{L(G)}(\lambda) = (\lambda + 2)^{m-n} Q_G(\lambda + 2),$$

where $Q_G(\lambda) = \det(\lambda I - Q)$ is the characteristic polynomial of the matrix Q .

The integral graphs are well studied in the literature. On the other hand, the graphs with integral Q -spectrum are studied in [4], [5], [8], [9] and [10], so far. Here we extend those results. Since Q -matrices are positive semidefinite the Q -spectrum consists of non-negative values. Furthermore, the largest eigenvalue of the signless Laplacian of a connected graph is a simple eigenvalue, while the least eigenvalue of the signless Laplacian of a connected graph is equal to 0 if and only if the graph is bipartite; in this case 0 is a simple eigenvalue (see [2], Proposition 2.1).

Recall that $L = D - A$ usually denotes the Laplacian matrix (of G). It is well known that if G is a bipartite graph then its Laplacian and signless Laplacian spectrum coincide (the proof can be found in many places, see [6], for example). In particular, a bipartite graph is L -integral if and only if it is Q -integral. Furthermore, since in this paper we deal with bipartite graphs only all results hold even if we consider Laplacian instead of signless Laplacian spectrum.

2. PRELIMINARIES

In this section we mention some results from literature in order to make the paper more self-contained. Recall that for an arbitrary edge of a graph G , the *edge-degree* is the number of edges adjacent to it. Also, we say that G is *edge-regular* if its edges have the same edge-degree. Further, an (r, s) -*semiregular bipartite* graph is a bipartite graph whose each vertex in the first (resp. second) colour class has degree r (resp. s). It is easy to check that a connected graph is edge-regular if and only if it is either regular or semiregular bipartite.

Following [8], [10] and [9], we list some results regarding Q -integral graphs. All Q -integral graphs with maximum edge-degree at most 4 are known; exactly 26 of them are connected. All Q -integral $(2, s)$ -semiregular bipartite graphs are determined (that is the infinite series) as well as all (r, s) -semiregular bipartite graphs with $r + s = 7$ and $r < 3 < s$; exactly 3 of them are connected. In addition, all possible Q -spectra of connected $(3, 4)$ and $(3, 5)$ -semiregular bipartite graphs are determined, and all the graphs having some of these Q -spectra are identified. Finally, all Q -integral graphs up to 10 vertices are known; exactly 172 of them are connected.

Now we list some notions and results to be used later on (see [2], Section 4; especially Theorem 4.1 and Corollaries 4.2 and 4.3). A $\frac{1}{2}$ -*edge walk* (of length k) in a graph G is an alternating sequence $v_1, e_1, v_2, e_2, \dots, v_k, e_k, v_{k+1}$ of vertices v_1, v_2, \dots, v_{k+1} and edges e_1, e_2, \dots, e_k such that for any $i = 1, 2, \dots, k$ the vertices v_i and v_{i+1} are end-vertices (not necessarily distinct) of the edge e_i . Let Q be the

signless Laplacian of a graph G . Then the (i, j) -entry of the matrix Q^k is equal to the number of $\frac{1}{2}$ -edge walks starting at vertex i and terminating at vertex j . Let $T_k = \sum_{i=1}^n \mu_i^k, (k = 0, 1, \dots)$ be the k -th spectral moment for the Q -spectrum (here $\mu_1, \mu_2, \dots, \mu_n$ are the Q -eigenvalues of G). Then T_k is equal to the number of closed $\frac{1}{2}$ -edge walks of length k . In particular, if G has n vertices, m edges, t triangles, and vertex-degrees d_1, d_2, \dots, d_n , then

$$T_0 = n, \quad T_1 = \sum_{i=1}^n d_i = 2m, \quad T_2 = 2m + \sum_{i=1}^n d_i^2, \quad T_3 = 6t + 3 \sum_{i=1}^n d_i^2 + \sum_{i=1}^n d_i^3.$$

Finally, if G is an (r, s) -semiregular bipartite graph which contains q quadrangles and h hexagons then for the spectral moments $T_k (k = 4, 5, 6)$ we have (cf. [8], Lemma 3.2)

$$\begin{aligned} T_4 &= (r^3 + s^3 + 4(r^2 + s^2) + 2(r + s) + 4rs - 2)m + 8q, \\ T_5 &= (r^4 + 5(r^3 + r^2 - r) + s^4 + 5(s^3 + s^2 - s) + 5rs(r + s + 2))m + 20(r + s)q, \\ T_6 &= (r^5 + s^5 + 6(r^4 + s^4) + 9(r^3 + s^3) - 7(r^2 + s^2) - 6(r + s + rs) + 6rs(r^2 + s^2 \\ &\quad + rs) + 21(r^2s + s^2r) + 4)m + 12(3(r^2 + s^2) + 2(r + s) + 4rs - 4)q + 12h. \end{aligned}$$

3. MAIN RESULTS

Before we proceed to the next lemma we emphasize the following formula. If G is a semiregular bipartite graph with n_1 (resp. n_2) vertices in the first (resp. second) colour class ($n_1 \geq n_2$), then the relation

$$(2) \quad P_{L(G)}(\lambda) = (\lambda - r + 2)^{n_1 - n_2} (\lambda + 2)^\ell \cdot \prod_{i=1}^{n_2} ((\lambda - r + 2)(\lambda - s + 2) - \lambda_i^2)$$

holds (compare [3], Proposition 1.2.18), where $\lambda_1, \lambda_2, \dots, \lambda_{n_2}$ are the first n_2 largest eigenvalues of G , while each vertex of the first (resp. second) colour class has degree r (resp. s) and $\ell = n_1 r_1 - n_1 - n_2$.

Lemma 3.1. *Let G be an (r, s) -semiregular bipartite graph with n_1 (resp. n_2) vertices in the first (resp. second) colour class ($n_1 \geq n_2$). Then its Q -spectrum lies in $[0, r] \cup [s, r + s]$.*

Proof. First, $n_1 \geq n_2$ implies $r \leq s$. Further, having in mind relations (2) and (1) we get

$$(3) \quad Q_G(\lambda) = (\lambda - r)^{n_1 - n_2} \cdot \prod_{i=1}^{n_2} ((\lambda - r)(\lambda - s) - \lambda_i^2),$$

where $\lambda_i (i = 1, 2, \dots, n_2)$ are defined above. Since the Q -eigenvalues are real, we have $(\lambda - r)(\lambda - s) \geq 0$, and the proof follows. \square

Note that formula (3) appears in [4], as well. By putting $\lambda_1 = \sqrt{rs}$ into (3), we get the least and the largest Q -eigenvalue: 0 and $r + s$, respectively. It is easy to see that the spectrum of G contains at least $n_1 - n_2$ eigenvalues equal to 0. The Q -eigenvalue r appears whenever G is not a regular graph, while the Q -eigenvalue s appears whenever the eigenvalue 0 has multiplicity strictly greater than $n_1 - n_2$ (in the spectrum of G).

Denote by $\text{mult}_Q(\mu)$ the multiplicity of the signless Laplacian eigenvalue μ of an arbitrary graph G . Now we have the following lemma.

Lemma 3.2. *Let G be a connected (r, s) -semiregular bipartite graph with n_1 (resp. n_2) vertices in the first (resp. second) colour class ($n_1 > n_2$). Let $\mu_1 = 0 < \mu_2 < \dots < \mu_k = r + s$ be the distinct Q -eigenvalues of G different from r and s . Then we have $\text{mult}_Q(\mu_i) = \text{mult}_Q(\mu_{k+1-i})$ ($i = 1, 2, \dots, \frac{k}{2}$). In addition, if α denotes the multiplicity of zero in the (adjacency) spectrum of G , then $\text{mult}_Q(r) = \frac{\alpha + n_1 - n_2}{2}$ and $\text{mult}_Q(s) = \frac{\alpha - n_1 + n_2}{2}$ hold.*

Proof. Since the spectrum of a bipartite graph is symmetric about zero, one can easily conclude that k and $\alpha \pm n_1 \mp n_2$ are even numbers. Further, the equalities $\text{mult}_Q(\mu_i) = \text{mult}_Q(\mu_{k+1-i})$ ($i = 1, 2, \dots, \frac{k}{2}$) follow from (3).

Finally, since $\alpha \geq n_1 - n_2$ holds, exactly $\frac{\alpha - n_1 + n_2}{2}$ of n_2 largest eigenvalues of G are equal to zero. Substituting $\lambda_i = 0$ ($i = \frac{\alpha - n_1 + n_2}{2}, \dots, n_2$) into (3), we get $\text{mult}_Q(r) = \frac{\alpha + n_1 - n_2}{2}$ and $\text{mult}_Q(s) = \frac{\alpha - n_1 + n_2}{2}$. The proof is complete. \square

The following simple consequence deserves the attention.

Corollary 3.1. *The Q -spectrum of an (r, s) -semiregular bipartite graph with the Q -eigenvalues r and s excluded is symmetric about $\frac{r+s}{2}$.*

Proof. The proof follows from the previous lemma and the relation (3). \square

In the following theorems we consider the Q -spectra of connected Q -integral $(3, s)$ -semiregular bipartite graphs. Since all such graphs with $s \leq 3$ are determined, while the cases $s = 4$ and $s = 5$ are considered in [8] and [9] (see also the previous section), we can assume that $s > 5$. From Lemma 3.1 we get that such graphs could have the following Q -eigenvalues: $0, 1, \dots, r, s, 1 + s, \dots, r + s$. We have $\text{mult}_Q(0) = \text{mult}_Q(r + s) = 1$. Due to Lemma 3.2, we can denote $\text{mult}_Q(i) = \text{mult}_Q(r + s - i) = a_i$ ($i = 1, 2, \dots, r - 1$) and $\text{mult}_Q(s) = a_s, \text{mult}_Q(r) = n_1 - n_2 + a_s$.

Theorem 3.1. *Let G be a connected Q -integral $(3, s)$ -semiregular bipartite graph ($s > 5$) with n vertices, m edges, q quadrangles and h hexagons. Then the following*

equalities hold (we use the notation above):

$$(4) \quad a_2 = \frac{(s-1)(s-4)a_1 + 3s(s-1) - 2q}{2(s-2)},$$

$$(5) \quad a_s = \frac{a_1 + q - 3}{s-2} - \frac{q}{s} - 1,$$

$$m = \frac{s(a_1(s-1) + 3(s+1))}{2},$$

$$h = \frac{s(s-1)((s-7)a_1 + 9(s+1)) - 2(s+7)q}{6}.$$

Proof. We get:

$$(6) \quad n_1 = \frac{m}{3}, \quad n_2 = \frac{m}{s}, \quad n = n_1 + n_2 = \frac{(3+s)m}{3s},$$

where n_1 (resp. n_2) is the number of vertices of degree 3 (resp. s). By using the formulas for the spectral moments T_k ($k = 0, 1, \dots, 6$) (see Section 2) and having in mind the equalities (6), we arrive at the following system of Diophantine equations:

$$2a_1 + 2a_2 + \frac{(s-3)m}{3s} + 2a_s + 2 = \frac{(s+3)m}{3s},$$

$$(s+3)a_1 + (s+3)a_2 + 3\left(\frac{(s-3)m}{3s} + a_s\right) + sa_s + 3 + s = 2m,$$

$$(1 + (s+2)^2)a_1 + (4 + (s+1)^2)a_2 + 9\left(\frac{(s-3)m}{3s} + a_s\right) + s^2a_s + (3+s)^2 = (5+s)m,$$

$$(1 + (s+2)^3)a_1 + (8 + (s+1)^3)a_2 + 27\left(\frac{(s-3)m}{3s} + a_s\right) + s^3a_s + (3+s)^3 = (18 + s^2 + 3s)m,$$

$$(1 + (s+2)^4)a_1 + (16 + (s+1)^4)a_2 + 81\left(\frac{(s-3)m}{3s} + a_s\right) + s^4a_s + (3+s)^4 = (67 + s^3 + 4s^2 + 14s)m + 8q,$$

$$(1 + (s+2)^5)a_1 + (32 + (s+1)^5)a_2 + 243\left(\frac{(s-3)m}{3s} + a_s\right) + s^5a_s + (3+s)^5 = (246 + s^4 + 5s^3 + 20s^2 + 70s)m + 20(3+s)q,$$

$$(1 + (s+2)^6)a_1 + (64 + (s+1)^6)a_2 + 729\left(\frac{(s-3)m}{3s} + a_s\right) + s^6a_s + (3+s)^6 = (895 + s^5 + 6s^4 + 27s^3 + 110s^2 + 327s)m + 12(29 + 3s^2 + 14s)q + 12h.$$

Solving this system for the variables a_2, a_s, m and h , we get the equalities above. \square

In the following theorems we consider some special cases of Theorem 3.1. We shall need the following well known result: if $\lambda_1 (= r), \lambda_2, \dots, \lambda_k$ are all the distinct eigenvalues of an arbitrary regular graph G (of vertex-degree r) on n vertices, then $(r - \lambda_2) \cdots (r - \lambda_k)$ is an integer divisible by n . (This fact can be proved by considering so-called Hoffman polynomial – see [7]).

Theorem 3.2. *Let G be a connected Q -integral $(3, s)$ -semiregular bipartite graph ($s > 5$) avoiding 1 in the Q -spectrum. Then its Q -spectrum has one of the forms given in Table 1. Each row contains the number of vertices (n), the number of edges*

(m), the multiplicities of the eigenvalues (a_2, a_s), the number of quadrangles (q) and the number of hexagons (h).

Proof. First, from Lemma 3.2 we have that the multiplicities of the Q -eigenvalues 1 and $2 + s$ are equal. So, according to notation introduced we can write $a_1 = 0$. It remains to compute the multiplicities a_2 and a_s .

Considering the system of Diophantine equations from the previous theorem, we get $a_1 = (s - 2)a_s + s - \frac{2q}{s} - 1$. Thus, $2q \equiv_s 0$ must hold, so we can write $q = \frac{ts}{2}$, where t is a non-negative integer. By putting $q = \frac{ts}{2}$ into (4) we get $a_2 = \frac{(t+3)s - 3s^2}{4 - 2s}$. Since a_2 is a non-negative number, we get $t \leq 3(s - 1)$ (recall that $s > 5$). Similarly, by putting $q = \frac{ts}{2}$ into (5) we get $a_s = \frac{t - s - 1}{s - 2}$. Having in mind that a_s is also non-negative, we get $t \geq s + 1$. By computing, we check that the other two variables (m and h) are non-negative for $q = \frac{ts}{2}$, $s + 1 \leq t \leq 3(s - 1)$. Moreover, since a_s is an integer as well, we get exactly three possible values of t . Namely, $t \in \{s + 1, 2s - 1, 3(s - 1)\}$.

n	m	a_2	a_s	q	h
$\frac{(s+1)(s+3)}{3}$	$s(s+1)$	s	0	$\frac{s(s+1)}{2}$	$\frac{4(s-2)s(s+1)}{3}$
22	48	4	1	68	456
51	126	7	1	203	812
210	576	16	1	1040	36000
$s + 3$	$3s$	0	2	$\frac{3(s-1)s}{2}$	$(s-2)(s-1)s$

Table 1

By computing the other values for $t = s + 1$ and $t = 3(s - 1)$, we obtain the values as in the first and the last row of Table 1, respectively. Note that all the values of the first row are integral if and only if $(s + 2) \not\equiv 0 \pmod{3}$ holds.

If $t = 2s - 1$, we get $n = \frac{(s+3)(s+4)}{6}$, $m = \frac{s(s+4)}{2}$. Since n and m are integers, we get that

$$(7) \quad (s - 2)s \equiv 0 \pmod{6}$$

must hold. Recall that line graph of a connected $(3, s)$ -semiregular bipartite graph with m edges is a connected regular graph on m vertices. By using the result mentioned above this theorem, we find that m divides $6s(s+1)(s+3)$ (the distinct Q -eigenvalues are $s + 3, s + 1, s, 3, 2$ and 0). In other words, $12s(s+1)(s+3) \equiv 0 \pmod{s(s+4)}$ must hold, and consequently we have $12(s+1) \equiv 0 \pmod{(s+4)}$. Using (7) we get that exactly one of numbers $(s - 2)$ and s is divisible by 6. If $s \equiv 0 \pmod{6}$ we get that $12(s+1) \equiv 0 \pmod{(s+4)}$ does not hold for any $s > 5$. If $(s - 2) \equiv 0 \pmod{6}$, $s > 5$, we get that $12(s+1) \equiv 0 \pmod{(s+4)}$ hold if and

only if $s \in \{8, 14, 32\}$, i.e. $t \in \{15, 27, 63\}$. By computing the other values for every possible t , we obtain the values as in the remaining rows of Table 1. \square

Now we consider some of signless Laplacian spectra of Table 1 (theoretically and by computer search). The results obtained are summarized in the following theorem.

Theorem 3.3. *There are exactly four graphs with data corresponding to the first row of Table 1 for $s = 6$ (H_1, \dots, H_4); there does not exist a graph with data corresponding to the second row. Finally, the only graphs with data corresponding to the last row are the complete bipartite graphs $K_{3,s}$. In the list below each vertex of the first colour class is represented by list of its neighbours (the vertices of the second colour class are labelled by numbers $1, 2, \dots, n_2$).*

Proof. By using the computer search we find the graphs H_1, \dots, H_4 . In the same way, we consider the second row. Finally, it is easy to prove that a connected $(3, s)$ -semiregular bipartite graph on $s + 3$ vertices and $3s$ edges must be $K_{3,s}$. In addition, this graph has the remaining data of the last row. \square

$$\begin{aligned}
 H_1 : & \left[\begin{array}{c|c|c|c|c|c|c} 1 & 2 & 3 & 1 & 3 & 6 & 1 & 4 & 7 & 2 & 3 & 4 & 2 & 5 & 6 & 3 & 4 & 5 & 3 & 6 & 7 \\ \hline 1 & 2 & 7 & 1 & 4 & 5 & 1 & 5 & 6 & 2 & 4 & 6 & 2 & 5 & 7 & 3 & 5 & 7 & 4 & 6 & 7 \end{array} \right] \\
 H_2 : & \left[\begin{array}{c|c|c|c|c|c|c} 1 & 2 & 3 & 1 & 3 & 7 & 1 & 4 & 7 & 2 & 3 & 4 & 2 & 5 & 7 & 3 & 4 & 5 & 3 & 6 & 7 \\ \hline 1 & 2 & 6 & 1 & 4 & 5 & 1 & 5 & 6 & 2 & 4 & 6 & 2 & 5 & 7 & 3 & 5 & 6 & 4 & 6 & 7 \end{array} \right] \\
 H_3 : & \left[\begin{array}{c|c|c|c|c|c|c} 1 & 2 & 3 & 1 & 3 & 5 & 1 & 4 & 7 & 2 & 3 & 4 & 2 & 5 & 7 & 3 & 4 & 5 & 3 & 6 & 7 \\ \hline 1 & 2 & 6 & 1 & 4 & 7 & 1 & 5 & 6 & 2 & 4 & 6 & 2 & 5 & 7 & 3 & 6 & 7 & 4 & 5 & 6 \end{array} \right] \\
 H_4 : & \left[\begin{array}{c|c|c|c|c|c|c} 1 & 2 & 3 & 1 & 4 & 5 & 1 & 6 & 7 & 2 & 4 & 7 & 2 & 5 & 6 & 3 & 4 & 6 & 3 & 5 & 7 \\ \hline 1 & 2 & 3 & 1 & 4 & 5 & 1 & 6 & 7 & 2 & 4 & 7 & 2 & 5 & 6 & 3 & 4 & 6 & 3 & 5 & 7 \end{array} \right]
 \end{aligned}$$

Theorem 3.4. *Let G be a connected Q -integral $(3, s)$ -semiregular bipartite graph ($s > 5$) with no induced quadrangles nor hexagons. Then it (if exists) has 504 vertices, 1008 edges, and the following Q -spectrum: $[9, 8^{63}, 7^{90}, 6^{14}, 3^{182}, 2^{90}, 1^{63}, 0]$ (the exponents stand for the multiplicities of the eigenvalues).*

Proof. Substituting $q = h = 0$ into the system of Diophantine equations given in the proof of Theorem 3.1, and solving it for the variables a_1, a_2, a_s and m , we get

$$\begin{aligned}
 a_1 &= \frac{9(s+1)}{7-s} & a_2 &= \frac{3(s-3)(s-1)(s+2)}{-s^2+9s-14} \\
 a_s &= \frac{(s+1)(s+2)}{-s^2+9s-14} & m &= \frac{3s(s+1)(s+2)}{7-s}
 \end{aligned}$$

Since all these variables are non-negative we get $s < 7$. By computing the other values for $s = 6$, we obtain the required values. \square

The existence of the graphs from the previous theorem as well as the remaining Q -spectra of Table 1 could be considered in forthcoming research.

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