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DISTANCE SPECTRUM AND ENERGY OF GRAPHS WITH SMALL DIAMETER

In honor of professor Dragoš Cvetković on the occasion of his 75th birthday

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In this paper we express the distance spectrum of graphs with small diameter in terms of the eigenvalues of their adjacency matrix. We also compute the distance energy of particular types of graph and determine a sequence of infinite families of distance equienergetic graphs.

1. Introduction

For a simple graph G with vertex set $V(G)$ and order $n = |V(G)|$, the characteristic polynomial Φ_G is defined as the characteristic polynomial of its adjacency matrix A_G. The *eigenvalues* of G,

$$\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G),$$

are then just the roots of Φ_G, and the *spectrum* of G, denoted by Σ_G, is the multiset of its eigenvalues.

If G is connected then its *distance matrix* is an $n \times n$ matrix $D_G = (d_{ij})$, where d_{ij} is the distance (length of a shortest path) between the vertices i and j. We denote the characteristic polynomial of D_G by Ψ_G. The *distance eigenvalues* (for short *D-eigenvalues*) of G are the roots of Ψ_G. They form the multiset called the *D-spectrum* of G.

The *energy* $\mathcal{E}(G)$ (resp. *distance energy* $\mathcal{DE}(G)$) is defined as the sum of the absolute values of the eigenvalues (resp. D-eigenvalues) of G. Two graphs are said

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to be *equienergetic* (resp. *distance equienergetic*) if their energies (resp. distance energies) coincide.

We present two general results that make a connection between the polynomials Ψ_G and Φ_G of a graph with diameter two and a bipartite semiregular graph with diameter three.

We also compute the D-spectrum of particular graphs with diameter at most four. For diameter two, these are graphs with exactly two main eigenvalues; for diameter three – bipartite semiregular graphs; for diameter four – bipartite regular incidence graphs of two-class symmetric partial incomplete block designs.

As a demonstration of these results, we determine the distance energy of certain families of graphs and obtain the conditions under which two graphs are distance equieneretic. The last is followed by a sequence of examples considering infinite families of distance equienergetic graphs.

Section 2 is preparatory. In Section 3, we compute the D-spectrum. Distance energy is considered in Section 4.

2. Preliminaries

We write K_n and $K_{m,n}$ for the complete graph with n vertices and the complete bipartite graph with m vertices in one and n vertices in the other colour class, respectively. For a graph G, we denote its complement by $\overline{G}$ and the disjoint union of k copies of G by kG. If $\mu_1, \mu_2, \ldots, \mu_k$ are all distinct eigenvalues of G, then we denote its spectrum by $\Sigma_G = \big\{[\mu_1]^{m_1}, [\mu_2]^{m_2}, \ldots, [\mu_k]^{m_k}\big\}$, where the exponents stand for the multiplicities of the eigenvalues.

Throughout the text we use I and J to denote identity and all-1 matrix, respectively. Their size will be clear from the context.

The subsequent definitions can be found in various literature, but here we recall them since the corresponding notation will be frequently used.

Suppose that the vertex set of a graph G can be partitioned into non-empty subsets $V_1, V_2, \ldots, V_s$ so that for any $i, j \in \{1, 2, \ldots, s\}$ each vertex in V_i is adjacent to exactly b_{ij} vertices of V_j. The multigraph H with adjacency matrix $B = (b_{ij})$ is called a *divisor* of G.

The eigenvalue of G is a *main eigenvalue* if and only if the corresponding eigenvector is not orthogonal to the all-1 vector. Otherwise, the corresponding eigenvalue is *non-main.* The divisor of a graph contains all main eigenvalues of G [**4**, Chapter 4]. It is also known that the largest eigenvalue of a connected graph is main [**15**].

The *main angles* of G are the cosines of the angles between the eigenspaces (the sets of eigenvectors belonging to an eigenvalue along with the all-0 vector) of G and the all-1 vector.

A graph with constant vertex degree r is called *regular* or *r-regular.*

A graph is called *bipartite semiregular* if it is bipartite and the vertices belonging to the same colour class have equal degree. The parameters (m, n, r_1, r_2) of such a graph respectively denote the size of each colour class and the vertex degree in each of them.

A connected graph G with diameter d is called *distance-regular* if there exist integers a_i, b_i, c_i such that for all i $(0 \leq i \leq d)$ and all vertices u and v at distance i, there are precisely a_i neighbours of v at distance i from u, b_i neighbours of v at distance $i+1$ from u, and c_i neighbours of v at distance $i-1$ from u. Obviously, G is b_0-regular.

An r-regular graph with n vertices distinct from K_n and $\overline{K_n}$ is said to be *strongly regular* with parameters (n, r, e, f) if there exist non-negative integers e and f such that every two adjacent vertices have exactly e common neighbours and every two non-adjacent vertices have exactly f common neighbours. It is not difficult to see that any connected strongly regular graph is distance-regular [**2**].

Let V be a finite set and $\mathcal{B}$ be a collection of subsets of the same size of V. A pair $(V, \mathcal{B})$ is called a *block design.* Elements of V and $\mathcal{B}$ are called *points* and *blocks*, respectively. If $V = \{p_1, p_2, \cdots, p_v\}$, $\mathcal{B} = \{B_1, B_2, \cdots, B_b\}$, then the $v \times b$ *incidence matrix* $N = (n_{ij})$ of $(V, \mathcal{B})$, is defined as $n_{ij} = 1$ when $p_i \in B_j$, and $n_{ij} = 0$ when $p_i \notin B_j$.

When $v = b$, a design is said to be *symmetric.* The *dual* of the design $(V, \mathcal{B})$ with the incidence matrix N is the design $(V, \mathcal{B})^*$ whose incidence matrix is N^T. The adjacency matrix of the *incidence graph* G of the design $(V, \mathcal{B})$ is defined as

$$\begin{pmatrix} 0 & N \\ N^T & 0 \end{pmatrix}. \tag{1}$$

A *partially balanced incomplete block design with* h $(h \geq 1)$ *associate classes (h-class PBIBD)* is an arrangement of v points in b blocks of size t, such that:

(a) Each of the v points occurs in exactly r blocks, and no point appears more then once in any block.

(b) There exists a relationship of association between every pair of the v points, satisfying the following conditions:

- Any two points are either first, second, ..., or hth associates, and any pair of points which are sth associates occur together in exactly c_s blocks $(1 \leq s \leq h)$.
- Each point has n_s sth associates.
- For any pair of points which are sth associates the number of points that are simultaneously jth associates of the first, and kth associates of the second point is p^s_{jk} and this number is independent of the pair of points with which we start. Furthermore, $p^s_{jk} = p^s_{kj}$ $(j \neq k;\ 1 \leq s, j, k \leq h)$.

Here we mostly deal with two-class symmetric PBIBDs, and we see from the previous definition (for $h = 2$) that its incidence matrix N satisfies the equation

$$NN^T = rI + c_1A_H + c_2(J - I - A_H), \tag{2}$$

where A_H is the adjacency matrix of a strongly regular graph H, and $c_1 > c_2$ are the parameters determined in (b). In particular, any two-class symmetric PBIBD is closely related to two different graphs. The first is its incidence graph with the adjacency matrix (1) and the second is mentioned strongly regular graph H. We denote such a design by $\mathrm{SPBIBD}_{c_1,c_2}(H)$ and say that it *is based on* H.

If $h = 1$, then $c_1 = c_2$ and the corresponding design is known as a *balanced incomplete block design (BIBD)*. In addition, if a BIBD is symmetric, then $v = b$ and $r = t$, and thus it is usually said that such a BIBD has the parameters (v, r, c_1).

3. Computing the distance spectrum

In this section we express the distance eigenvalues in terms of the eigenvalues (of the adjacency matrix) of particular types of graph specified in the corresponding subsections. We start with the following general results that constitute the connection between the corresponding characteristic polynomials. Our Theorem 1 is a special case of a more general result (see [**6**], page 90), however for the convenience of the reader we present it with proof.

Theorem 1. *The characteristic polynomial of the distance matrix of a graph G with n vertices and diameter two is determined by*

$$\Psi_G(x) = (-1)^n\Phi_G(-x-2)\left(1 - 2n\sum_{i=1}^{k}\frac{\beta_i^2}{x+2+\mu_i}\right), \tag{3}$$

where μ_i and β_i $(1 \leq i \leq k)$ are all distinct main eigenvalues of the adjacency matrix and the corresponding main angles.

Proof Since the diameter of G is two, its distance matrix can be represented as $D_G = 2(J - I) - A_G$, and so we have $\Psi_G(x) = \det((x+2)I + A_G - 2J)$. Denote the matrix $(x + 2)I + A_G$ by B, and let for $S \subseteq \mathbb{N}_n$, B_S stand for the matrix obtained from B by replacing all its columns indexed by S with $(-2)\mathbf{j}$. Using the multilinearity of a determinant, we get

$$\det((x+2)I + A_G - 2J) = \sum_{S\subseteq\mathbb{N}_n}\det(B_S),$$

where the sum goes over 2^n summands. Since the corresponding determinant is zero whenever $|S| > 1$, the last term is equal to

$$\det(B)+\sum_{i=1}^{n}\det(B_i),$$

where we write B_i for $B_{\{i\}}$. Using the Laplacian development over the ith column of B_i, and writing $\widehat{B}_{ij}$ for the (i,j)-cofactor of B, we get

$$\sum_{i=1}^{n}\det(B_i)=-2\sum_{i=1}^{n}\sum_{j=1}^{n}(-1)^{i+j}\widehat{B}_{ij}=-2\mathbf{j}^T\mathrm{adj}(B)\mathbf{j}=-2\mathbf{j}^T\det(B)B^{-1}\mathbf{j}.$$

Setting this in the previous expression and using spectral decomposition of B (if necessary, see [**7**, p. 11]), we get

$$\Psi_G(x)=\det(B)\left(1-2\mathbf{j}^TB^{-1}\mathbf{j}\right)=(-1)^n\Phi_G(-x-2)\left(1-2\sum_{i=1}^{k}\frac{\mathbf{j}^TP_i\mathbf{j}}{x+2+\mu_i}\right),$$

where P_i represents the orthogonal projection of $\mathbb{R}^n$ onto the eigenspace of μ_i. The proof follows by $\mathbf{j}^TP_i\mathbf{j}=n\beta_i^2$. $\square$

It is known that a connected graph G has exactly one main eigenvalue if and only if G is regular [**15**]. Thus, for a connected r-regular graph G of diameter two, we have

$$\Psi_G(x)=(-1)^n\Phi_G(-x-2)\left(1-\frac{2n}{x+2+r}\right).$$

The last equality is also obtained in [**12**]. Connected graphs with exactly two main eigenvalues are considered in forthcoming Subsection 3.1.

Imitating the previous proof, we determine the characteristic polynomial of the distance matrix of a bipartite semiregular graph with diameter three.

Theorem 2. *The characteristic polynomial of the distance matrix of a bipartite semiregular graph G with parameters (m,n,r_1,r_2) and diameter three is determined by*

$$\begin{aligned}\Psi_G(x) &= (-1)^{m+n}\frac{\Phi_G\left(-\frac{1}{2}(x+2)\right)}{\frac{1}{4}(x+2)^2-r_1r_2}\cdot\\ &\quad\cdot\left(x^2-2(m+n-2)x-5mn-4(m+n-r_1r_2+3r_1m+1)\right).\end{aligned}\tag{4}$$

Proof The distance matrix can be represented as $D_G=2J-2I-2A_G+A_{K_{m,n}}$, and thus we have $\Psi_G(x)=\det\left((x+2)I+2A_G-A_{K_{m,n}}-2J\right)$. Denoting the matrix

$(x+2)I + 2A_G - A_{K_{m,n}}$ by B, and using the fact that the matrices I, A_G, and $A_{K_{m,n}}$ commute, we arrive at

$$\det(B) = (-1)^{m+n}(x+2+(2\lambda_1-\sqrt{mn}))(x+2-(2\lambda_1-\sqrt{mn}))\frac{\Phi_G\left(-\frac{1}{2}(x+2)\right)}{\frac{1}{4}(x+2)^2 - r_1r_2}.$$

Assume that $r_1 \neq r_2$. Recall that every bipartite semiregular graph has two main eigenvalues $\pm\sqrt{r_1r_2}$ [**15**, Proposition 3.5]. The eigenspaces of the eigenvalues $\pm\sqrt{r_1r_2}$ of A_G coincide with the eigenspaces of the eigenvalues $\pm\sqrt{mn}$ of $A_{K_{m,n}}$, and so we have

$$\begin{aligned} \Psi_G(x) &= \det(B)\left(1-2\mathbf{j}^TB^{-1}\mathbf{j}\right) \\ &= \det(B)\left(1-2\left(\frac{\mathbf{j}^TP_1\mathbf{j}}{x+2+2\lambda_1-\sqrt{mn}}+\frac{\mathbf{j}^TP_2\mathbf{j}}{x+2-2\lambda_1+\sqrt{mn}}\right)\right), \end{aligned} \tag{5}$$

where P_1 (resp. P_2) represents the orthogonal projection of $\mathbb{R}^n$ onto the eigenspace of $\sqrt{r_1r_2}$ (resp. $-\sqrt{r_1r_2}$). The assertion follows by $\mathbf{j}^TP_1\mathbf{j} = \frac{m+n}{2} + \sqrt{mn}$ and $\mathbf{j}^TP_2\mathbf{j} = \frac{m+n}{2} - \sqrt{mn}$.

The special case $r_1 = r_2$ also implies $m = n$, while G reduces to a regular graph with the unique main eigenvalue $\lambda_1 = r_1$. The last summand in (5) is lost, and the assertion follows by $\mathbf{j}^TP_1\mathbf{j} = 2n$. □

We give a deeper analyze of the last result in Subsection 3.2.

3.1 Diameter 2: Graphs with exactly two main eigenvalues

Here is a corollary of Theorem 1.

Corollary 1. *Let G be a connected graph with n vertices, m edges, diameter two, and exactly two main eigenvalues λ_1 and λ_k. Then the D-eigenvalues of G are $-\lambda_i - 2$ $(i \notin \{1,k\})$ and the roots of the quadratic equation*

$$x^2 + (\lambda_1+\lambda_k-2n+4)x + \lambda_1\lambda_k - 2(\lambda_1+\lambda_k)(n-1) + 4(m-n+1) = 0.$$

Proof Denote by β_1 and β_k the angles corresponding to the eigenvalues λ_1 and λ_k, respectively. Then $\beta_1^2 + \beta_k^2 = 1$, and also $\lambda_1\beta_1^2 + \lambda_k\beta_k^2 = \frac{2m}{n}$ [**15**, Proposition 1.2]. The proof follows after eliminating β_1 and β_k in the equation (3). □

Graphs with diameter two and exactly two main eigenvalues include non-regular complete bipartite graphs, the complement of the disjoint union of an arbitrary number of such graphs or graphs obtained by removing an arbitrary vertex from a strongly regular graph (with last parameter at least two) [**15**]. More examples of such graphs can be found in [**3**].

3.2 Diameter 3: Bipartite semiregular graphs

We can rephrase Theorem 2 as follows.

Corollary 2. *Let G be a bipartite semiregular graph G with parameters (m, n, r_1, r_2) and diameter three. If the eigenvalues of G are $\lambda_1 = \sqrt{r_1 r_2}, \lambda_2, \ldots, \lambda_{m+n} = -\sqrt{r_1 r_2}$, then its D-eigenvalues are $-2\lambda_i - 2$ $(2 \leq i \leq m + n - 1)$ and*

$$m + n - 2 \pm \sqrt{m^2 + n^2 + 7mn + 4r_1 r_2 - 12r_1 m}.$$

If, in addition, G is regular then we have the following assertion.

Corollary 3. *The characteristic polynomial of the distance matrix of a bipartite r-regular graph with $2n$ vertices and diameter three is determined by*

$$\Psi_G(x) = \frac{\Phi_G\left(-\frac{1}{2}(x+2)\right)}{\frac{1}{4}(x+2)^2 - r^2}\left(x^2 - 4(n-1)x - 8n - 5n^2 - 4r^2 + 12rn + 4\right). \tag{6}$$

In other words, if the eigenvalues of G are $\lambda_1 = r, \lambda_2, \ldots, \lambda_{2n} = -r$, then its D-eigenvalues are $-2\lambda_i - 2$ $(2 \leq i \leq 2n - 1)$, $5n - 2r - 2$, and $2r - n - 2$.

If a graph considered in this corollary has exactly three distinct D-eigenvalues, then observe that it must have exactly four distinct eigenvalues. Namely, since diameter is three the number of distinct eigenvalues is at least four (cf. [**4**, Theorem 3.13]), and if it is five or more, then simple connection between eigenvalues and D-eigenvalues together with bipartiteness contradicts the existence of exactly three distinct D-eigenvalues.

This observation enables us to improve a result reported in our previous work [**1**]. It has been shown that a bipartite distance-regular graph with diameter three has exactly three distinct D-eigenvalues if and only if it is the incidence graph of a symmetric BIBD with parameters $(4s^2, 2s^2 + s, s^2 + s)$. Such designs are called Menon designs. Now, according to the above discussion, if a bipartite regular graph with diameter three has exactly three distinct D-eigenvalues, then it has exactly four distinct eigenvalues, but then it must be an incidence graph of a symmetric BIBD (see [**4**], page 166), and by Theorem 1.6.1 of [**2**], it is then distance-regular. According to this, we may reduce the distance-regularity condition to (simple) regularity, and so bearing in mind the proof of [**1**, Theorem 3.4], we may state the following result.

Corollary 4. *A bipartite regular graph with diameter three has exactly three distinct D-eigenvalues if and only if it is the incidence graph of a Menon design.*

Another interesting fact is that there exist bipartite regular graphs with diameter three that are not distance-regular, but have exactly four distinct D-eigenvalues. (In the case of the spectrum of the adjacency matrix, this is not true. As mentioned above every bipartite regular graph with diameter three and exactly four distinct eigenvalues of the adjacency matrix must be distance-regular). For example, there are six non-isomorphic bipartite 6-regular graphs with 24 vertices and the spectrum $\{[\pm 6]^1, [\pm 2]^9, [0]^4\}$ (see [**13**, Table 1]). Each of them is the incidence graph of the $\mathrm{SPBIBD}_{3,2}(\overline{3K_4})$, so its diameter is three. By Corollary 3, their common D-spectrum is $\{[46]^1, [2]^9, [-2]^5, [-6]^9\}$.

3.3 Diameter 4: Bipartite regular incidence graphs of two-class symmetric PBIBDs

Let G be a bipartite regular incidence graph of a two-class symmetric PBIBD based on a strongly regular graph H. If (1) is the adjacency matrix of G, then from (2) it follows that NN^T has three distinct eigenvalues. Thus, G has either six or five (if one of the eigenvalues of NN^T equals zero) distinct eigenvalues. If the diameter of G is four, then the second parameter c_2 is zero, and so according to the notation introduced in Section 2, we may briefly say that G is the incidence graph of a $\mathrm{SPBIBD}_{c_1,0}(H)$. In general case, only what we can say about the dual of $\mathrm{SPBIBD}_{c_1,0}(H)$ is that it is some block design, but under the additional assumption we may give the following result.

Theorem 3. *Let G be a bipartite r-regular graph with $2n$ vertices and diameter four. If G is the incidence graph of $SPBIBD_{c_1,0}(H)$ and if $SPBIBD_{c_1,0}(H)^*$ is also a two-class symmetric PBIBD with the same parameters then:*

1. *If the spectrum of G is $\{\pm r, [\pm\lambda_2]^{m_2}, [0]^{m_3}\}$, then its D-spectrum is* $\Big\{7n - 4 - 2r - \frac{2}{c_1}(r^2 - r), n + 2r - 4 - \frac{2}{c_1}(r^2 - r), \big[\frac{2r}{c_1} - \frac{2\lambda_2^2}{c_1} - 4 \pm 2\lambda_2\big]^{m_2}, \big[\frac{2r}{c_1}\big]^{m_3}\Big\}$.
2. *If the spectrum of G is $\{\pm r, [\pm\lambda_2]^{m_2}, [\pm\lambda_3]^{m_3}\}$, then its D-spectrum is* $\Big\{7n - 4 - 2r - \frac{2}{c_1}(r^2 - r), n + 2r - 4 - \frac{2}{c_1}(r^2 - r), \big[\frac{2r}{c_1} - \frac{2\lambda_2^2}{c_1} - 4 \pm 2\lambda_2\big]^{m_2}, \big[\frac{2r}{c_1} - \frac{2\lambda_3^2}{c_1} - 4 \pm 2\lambda_3\big]^{m_3}\Big\}$.

Proof Consider the blocking of the distance matrix of G induced by the colour classes

$$D_G = \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix}.$$

Then the adjacency matrix of G has the form (1), and so we immediately get

$$D_2 = 3J - 2N \quad \text{and} \quad D_3 = 3J - 2N^T.$$

Next, since G is the incidence graph of $\mathrm{SPBIBD}_{c_1,0}(H)$, by (2) we have $NN^T = rI + c_1 A_H$, and similarly since $\mathrm{SPBIBD}_{c_1,0}(H)^*$ is also a two-class symmetric PBIBD with the same parameters, we have $N^T N = rI + c_1 A_F$, where A_F is the adjacency matrix of some strongly regular graph F. Observe now that two vertices belonging to the same colour class are at distance 2 in G if the corresponding element of NN^T (or N^TN) is c_1, or at distance 4 in G if this element is 0. Hence, using the the above expressions for NN^T and N^TN, we get

$$D_1 = \frac{2}{c_1}(NN^T - rI) + 4\left(J - I - \frac{1}{c_1}(NN^T - rI)\right)$$

and

$$D_4 = \frac{2}{c_1}(N^TN - rI) + 4\left(J - I - \frac{1}{c_1}(N^TN - rI)\right).$$

It follows that the distance matrix D may be rewritten as

$$D_G = 4J - A_{K_{n,n}} - \left(4 - \frac{2r}{c_1}\right)I - 2A_G - \frac{2}{c_1}A_G^2.$$

Since all the matrices in this equality commute, the result follows by direct computation. $\square$

It is shown in [**9**] that, under certain assumptions, the dual of an h-class symmetric PBIBD is again an h-class symmetric PBIBD with the same parameters. In the same paper, the author constructed the $\mathrm{SPBIBD}_{2,0}(\overline{4K_2})$ whose dual is not a PBIBD. The corresponding incidence graph is known as the Hoffman graph.

Every bipartite distance-regular graph with diameter four is the incidence graph of a $\mathrm{SPBIBD}_{c_1,0}(H)$, where H is one, so-called, halved graph of G [**14**, Proposition 4.1]. Observe that the dual $\mathrm{SPBIBD}_{c_1,0}(H)^*$ is also a two-class symmetric PBIBD with the same parameters based on the other halved graph of G (note that c_1 is in fact the parameter c_2 from the intersection array of G). Thus, the previous theorem can be applied to establish the connection between the distance eigenvalues and the eigenvalues of the adjacency matrix of a given bipartite distance-regular graph with diameter four.

4. Distance energy and distance equienergetic graphs

Throughout the following two examples, we use Corollary 1 to determine the distance energy of some graphs with diameter two. Recall from [**5**] that the automorphism group of a graph G has at least s orbits in $V(G)$, where s denotes the number of main eigenvalues of G, and that the orbits of the automorphism group of G form its equitable partition. It follows that any non-regular graph with a divisor of order two has precisely two main eigenvalues, which are simultaneously the eigenvalues of the divisor.

Example 1: Consider the graph $G = \overline{pK_{a,b}}$ ($a \neq b$, $p > 1$). Its diameter is two, while the vertices can be partitioned, in a natural way, into two subsets A and

B of size pa and pb, respectively, where the subgraph induced by the vertices in A (resp. B) is complete, and every vertex belonging to A (resp. B) has $(p-1)b$ (resp. $(p-1)a$) neighbours in B (resp. A). Thus, its main eigenvalues are the eigenvalues of the divisor matrix

$$\begin{pmatrix} pa-1 & (p-1)b \\ (p-1)a & pb-1 \end{pmatrix}.$$

The (disconnected) graph $\overline{G} = pK_{a,b}$ has the spectrum $\{[\sqrt{ab}]^p, [0]^{p(a+b-2)}, [-\sqrt{ab}]^p\}$. Using the relations between the eigenvalues of complementary graphs (see [**8**, Theorem 4] and [**4**, Theorem 2.5]) and Corollary 1, we compute the D-eigenvalues of G. They are: -1 with multiplicity $p(a+b-2)$, $\pm\sqrt{ab}-1$ with multiplicity $p-1$, and $\frac{1}{2}\left(p(a+b) \pm \sqrt{p^2(a+b)^2+4ab(2p+1)}\right)-1$. The last two D-eigenvalues are of the opposite sign, so the distance energy of G is

$$\mathcal{DE}(G) = 2\left((p-1)(\sqrt{ab}-1) + \frac{1}{2}\left(p(a+b)+\sqrt{p^2(a+b)^2+4ab(2p+1)}\right)-1\right).$$

□

Example 2: If H is a bipartite r-regular graph with $2n$ vertices, diameter three, and the adjacency matrix (1), then the graph G with the adjacency matrix

$$\begin{pmatrix} J-I & N \\ N^T & 0 \end{pmatrix},$$

has diameter two, and the matrix

$$\begin{pmatrix} n-1 & r \\ r & 0 \end{pmatrix}$$

belongs to its divisor. Thus, G has two main eigenvalues, which are roots of the quadratic equation $x^2-(n-1)x-r^2=0$. The characteristic polynomial of G is

$$\Phi_G(x) = \begin{vmatrix} J-I-xI & N \\ N^T & -xI \end{vmatrix}.$$

We easily compute $\Phi_G(x) = \det\left(-xJ+(x^2+x)I-N^TN\right)$, and since the determinant of a matrix is the product of its eigenvalues, we get $\Phi_G(x) = (x^2-(n-1)x-r^2)\prod_{\lambda_i\in\Sigma_H,\lambda_i\neq\pm r}(x^2+x-\lambda_i^2)$, where Σ_H denotes the spectrum of H. By Corollary 1, the D-eigenvalues of G are $\frac{1}{2}\left(3(n-1)\pm\sqrt{17n^2-16nr-2n+4r^2+1}\right)$, and $\frac{1}{2}\left(-3\pm\sqrt{1+4\lambda_i^2}\right)$, where λ_i are the eigenvalues of H distinct from $\pm r$.

If $\sigma = \sum_{\lambda_i\in\Sigma_H,\lambda_i\neq\pm r,|\lambda_i|>\sqrt{2}}\left(\sqrt{1+4\lambda_i^2}-3\right)$, then the distance energy of G is given by

$$\mathcal{DE}(G) = 6n-6+\sigma$$

or

$$\mathcal{DE}(G) = 3n - 3 + \sqrt{17n^2 - 16nr - 2n + 4r^2 + 1} + \sigma,$$

depending on whether two main D-eigenvalues are of the same or the opposite sign. $\square$

Here are the two auxiliary results concerning distance equienergetic graphs.

Lemma 1. *Let G_1 and G_2 be bipartite regular graphs of degree r_1 and r_2 respectively, with $2n$ vertices and diameter three. Assume that $2r_i - n - 2 \geq 0$ holds for $1 \leq i \leq 2$. If Σ_1 and Σ_2 respectively denote the spectra of G_1 and G_2, then G_1 and G_2 are distance equienergetic if and only if*

$$\sum_{\substack{\lambda_i \in \Sigma_1,\\ \lambda_i < -1,\\ \lambda_i \neq -r_1}} (1 + \lambda_i) = \sum_{\substack{\lambda_i \in \Sigma_2,\\ \lambda_i < -1,\\ \lambda_i \neq -r_2}} (1 + \lambda_i), \tag{7}$$

and in that case their common distance energy is

$$8n - 8 - 4 \sum_{\substack{\lambda_i \in \Sigma_1,\\ \lambda_i < -1,\\ \lambda_i \neq -r_1}} (1 + \lambda_i). \tag{8}$$

Proof The distance energy of any graph is twice the sum of its positive D-eigenvalues. Since $2r_i - n - 2 \geq 0$, using Corollary 3 we get that the distance energy of G_1 is given by (8), and similarly holds for the distance energy of G_2 (with Σ_2 and r_2 instead of Σ_1 and r_1).

Now, we have that $\mathcal{DE}(G_1) = \mathcal{DE}(G_2)$ if and only if (7) holds. $\square$

Lemma 2. *Let G_1 and G_2 be bipartite r-regular equienergetic graphs with $2n$ vertices and diameter three. If all eigenvalues of these graphs lie outside the interval $(-1, 1)$, then G_1 and G_2 are distance equienergetic and their common distance energy is $5n - 6r - 2 + |2r - n - 2| + 2\mathcal{E}(G_i)$.*

Proof Since G_1 and G_2 are equienergetic and of the same degree, we have that $\sum_{j=2}^{n} \lambda_j(G_1) = \sum_{j=2}^{n} \lambda_j(G_2)$. Using Corollary 3, we compute

$$\begin{aligned}
\mathcal{DE}(G_i) &= 5n-2r-2+|2r-n-2|+2\sum_{j=2}^{2n-1}|1+\lambda_j(G_i)| \\
&= 5n-2r-2+|2r-n-2|+2\left(\sum_{j=2}^{n}(1+\lambda_j(G_i))-\sum_{j=n+1}^{2n-1}(1+\lambda_j(G_i))\right) \\
&= 5n-6r-2+|2r-n-2|+2\mathcal{E}(G_i),
\end{aligned}$$

which yields $\mathcal{DE}(G_1)=\mathcal{DE}(G_2)$. □

We use Corollary 3 and the previous lemmas to give some examples of distance equienergetic bipartite regular graphs with diameter three.

Recall that the *bipartite complement* of a bipartite graph G with two colour classes U and W is the bipartite graph $\overline{\overline{G}}$ with the same colour classes having the edge between U and W exactly where G does not. By [**17**], the characteristic polynomials of G and $\overline{\overline{G}}$ satisfy

$$\frac{\Phi_G(x)}{x^2-r^2}=\frac{\Phi_{\overline{\overline{G}}}(x)}{x^2-(n-r)^2}. \tag{9}$$

Example 3: In [**16**], the authors showed that the distance energy of $K_{m,n}$ $(m,n\geq 2)$ is $4(m+n-2)$, and consequently, for every $n\geq 4$, they obtained the family of bipartite distance equienergetic graphs: $\left\{K_{2,n-2},K_{3,n-3},\dots,K_{\lfloor\frac{n}{2}\rfloor,\lceil\frac{n}{2}\rceil}\right\}$.

According to (9) and Corollary 3, we have that the distance energy of $\overline{\overline{pK_2}}$ $(p\geq 3)$ is $8(p-1)$. Thus, for every even $n\geq 6$, the above mentioned family also contains the graph $\overline{\overline{\frac{n}{2}K_2}}$. □

Example 4: Consider the graphs $G_1=\overline{\overline{mK_{n,n}}}$ and $G_2=\overline{\overline{nK_{m,m}}}$, where m and n are integers greater than 1. Their spectra are $\Sigma_1=\left\{\pm mn,[\pm n]^{m-1},[0]^{2m(n-1)}\right\}$ and $\Sigma_2=\left\{\pm mn,[\pm m]^{n-1},[0]^{2n(m-1)}\right\}$, respectively.

Their common order and vertex degree are $2mn$ and mn, so the first assumption of Lemma 1 is satisfied. Also, since $\sum_{\lambda_i\in\Sigma_1,\lambda_i<-1,\lambda_i\neq -mn}(1+\lambda_i)=(m-1)(1-n)$ and $\sum_{\lambda_i\in\Sigma_2,\lambda_i<-1,\lambda_i\neq -mn}(1+\lambda_i)=(n-1)(1-m)$, we conclude that the same holds for the second assumption, and therefore G_1 and G_2 are distance equienergetic with common distance energy $4(3mn-m-n-1)$. Note that the same result, but only for m and n being primes and in a different way, is obtained in [**11**] (the corresponding graphs are known as the integral circulant graphs). □

Example 5: Let G_1 be the graph from the previous example, but now let m and n be taken as the roots of the quadratic equation $x^2-(2p+q-1)x+pq=0$, for

$p \geq 2$, $q \geq 3$ integral. If H is the bipartite complement of $\overline{\overline{pqK_2}}$, then its diameter is three and the spectrum is $\Sigma_H = \big\{\pm((p-1)q+1), [\pm\ (q-1)]^{p-1}, [\pm 1]^{p(q-1)}\big\}$.

We are going to show that $\mathcal{DE}(G_1) = \mathcal{DE}(H)$. The first assumption of Lemma 1 is verified by direct computation. Since $\sum_{\lambda_i \in \Sigma_H, \lambda_i < -1, \lambda_i \neq -((p-1)q+1)} (1+\lambda_i) = (2-q)(p-1)$, we get that the second assumption holds if $(m-1)(1-n) = (2-q)(p-1)$, and this is true because m and n are the roots of the the above equation. So, G_1 and H are equienergetic, and their common distance energy is $4(3mn-m-n-1) = 4(3pq-2p-q)$.

It is easy to see that for $p \geq 3$ odd and $q = \frac{3p+1}{2}+1$ or $p \geq 4$ even and $q = \frac{3p}{2}-1$, both m and n are integral and greater than 1.

In fact, this construction produces distance equienergetic triplets, since by the previous example, G_1 is paired with a distance equienergetic graph distinct from H. □

For a graph G we denote by $L^2(G)$ the line graph of its line graph. Its *extended bipartite double*, $\mathrm{ebd}(G)$, is the graph with adjacency matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes (A_G + I).$$

The extended bipartite double of a graph G is bipartite, and it is connected if and only if G is connected. If G has the spectrum Σ_G, then $\mathrm{ebd}(G)$ has the spectrum $(-\Sigma_G - 1) \cup (\Sigma_G + 1)$ [**2**, Theorem 1.11.2].

Example 6: Let G_1 and G_2 be two regular graphs with n vertices and vertex degree $r \geq 4$.

Let $F_i = \mathrm{ebd}(\overline{L^2(G_i)})$ $(1 \leq i \leq 2)$. According to [**10**], these graphs are bipartite and equienergetic, with common order $nr(r-1)$ and vertex degree $\frac{nr(r-1)}{2} - 4r + 6$, while their spectra are given by $\Big\{\pm(\frac{nr(r-1)}{2} - 4r + 6), \pm(-\lambda_2(G_i) - 3r + 6), \ldots, \pm(-\lambda_n(G_i) - 3r + 6), [\pm(-2r+6)]^{\frac{n(r-2)}{2}}, [\pm 2]^{\frac{nr(r-2)}{2}}\Big\}$.

The order and vertex degree satisfy $2\left(\frac{nr(r-1)}{2} - 4r + 6\right) - nr(r-1) > 0$, which leads to the conclusion that any two vertices in the same colour class have a common neighbour, and consequently, the diameter of both graphs is three. Concerning the spectra we conclude that all eigenvalues of both graphs lie outside the interval $(-1, 1)$. Thus, by Lemma 2, they are distance equienergetic and their common distance energy is $2(5nr^2 - 9nr - 8r + 10)$. □

Example 7: Let G_1 and G_2 be two regular graphs with n vertices and vertex degree $r \geq 3$.

Let now $F_i = \overline{\overline{\mathrm{ebd}(L^2(G_i))}}$ $(1 \leq i \leq 2)$. According to [**10**], the graphs $\mathrm{ebd}(L^2(G_i))$ $(1 \leq i \leq 2)$ are bipartite and equienergetic with the common order $nr(r-1)$ and degree $4r-5$, while their spectra are $\Big\{\pm(4r-5), \pm(\lambda_2(G_i) + 3r - 5), \ldots, \pm(\lambda_n(G_i) + 3r - 5), [\pm(2r-5)]^{\frac{n(r-2)}{2}}, [\pm 1]^{\frac{nr(r-2)}{2}}\Big\}$.

It follows by (9) that F_1 and F_2 are also equienergetic with the common order $nr(r-1)$ and vertex degree $\frac{nr(r-1)-8r+10}{2}$. Following the previous example and using (9), we conclude that they are distance equienergetic with distance energy $4((nr+2)(2r-3)-8r+10)$. $\square$

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