

Research Article

An upper bound for the Laplacian index of a signed graph

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Abstract

We prove that

$$\mu_1 \leq \max \left\{ \sqrt{2(d_i^2 + d_i m_i - 2T_i^+)} : 1 \leq i \leq n \right\},$$

where μ_1 is the Laplacian index of a signed graph \dot{G} with n vertices and, for a vertex i , the symbols d_i , m_i and T_i^+ denote its degree, average 2-degree and the number of positive triangles containing i , respectively. We also show that equality holds if and only if \dot{G} is switching equivalent to a regular signed graph with all edges being negative. Apart from this result, we derive some other upper bounds for μ_1 , make some comparisons and conclude by finding a lower bound for the same eigenvalue.

Keywords: signed graph; largest eigenvalue; vertex degree; regular signed graph.

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1. Introduction

A signed graph \dot{G} is a pair (G, σ) , where $G = (V, E)$ is an unsigned graph, called the *underlying graph*, and $\sigma: E \rightarrow \{+1, -1\}$ is the *sign function*. The edge set of a signed graph is composed of subsets of positive and negative edges. We denote the number of vertices of a signed graph by n .

The $n \times n$ adjacency matrix $A_{\dot{G}}$ of \dot{G} is obtained from the standard $(0, 1)$ -adjacency matrix of G by reversing the sign of all 1s which correspond to negative edges. If $D_{\dot{G}}$ is the diagonal matrix of vertex degrees, then the *Laplacian matrix* of \dot{G} is defined as $L_{\dot{G}} = D_{\dot{G}} - A_{\dot{G}}$. The eigenvalues of $L_{\dot{G}}$ are real and non-negative, and they form the *Laplacian spectrum* of \dot{G} . The largest eigenvalue of $L_{\dot{G}}$ is called the *Laplacian index* and denoted by μ_1 .

Spectra of standard matrices associated with signed graphs have received a great deal of attention in the recent past. It occurs that, as in the particular case of graphs, the largest eigenvalue is the most investigated one. It gives information about the structure and has applications in various fields. Some details can be found in [5, 7, 8]. For some recently obtained upper bounds, we refer the reader to [3, 4].

Section 2 contains some definitions, additional terminology and notation. Our contribution is reported in Section 3. The main result is the sharp upper bound given at the beginning of that section. It is followed by some other bounds and their comparisons.

2. Preparatory

For U a subset of the vertex set, let \dot{G}^U be the signed graph obtained from \dot{G} by reversing the sign of each edge between a vertex in U and a vertex in $V(\dot{G}) \setminus U$. The signed graph \dot{G}^U is said to be *switching equivalent* to \dot{G} . The switching equivalence is an equivalence relation and switching equivalent signed graphs share the same Laplacian spectrum.

We denote by d_i the degree of a vertex $i \in V(\dot{G})$. We also write d_i^+ and d_i^- for the positive and negative vertex degree (i.e., the number of positive and negative edges incident with i). If the vertices i and j are adjacent, then we write $i \sim j$. The existence of a positive (resp. negative) edge between these vertices is designated by $i \overset{+}{\sim} j$ ($i \overset{-}{\sim} j$). For $d_i > 0$, we denote

$$m_i^+ = \frac{1}{d_i} \left(\sum_{j \overset{+}{\sim} i} d_j^+ + \sum_{j \overset{-}{\sim} i} d_j^- \right), \quad m_i^- = \frac{1}{d_i} \left(\sum_{j \overset{-}{\sim} i} d_j^- + \sum_{j \overset{+}{\sim} i} d_j^+ \right)$$

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and $m_i = m_i^+ + m_i^-$; the last parameter is known as the *average 2-degree*, that is the average degree of the neighbours of i . For $d_i = 0$, we define all these parameters to be zero.

For the vertices i and j , we use c_{ij} to denote the number of their common neighbours (i.e., joined to both of them by any edge), c_{ij}^+ for the number of common neighbours joined to i by a positive edge and to j by any edge, and c_{ij}^{++} for the number of common neighbours that are joined to both of them by a positive edge. We also use the similar notation for all the remaining possibilities.

A cycle in \dot{G} is said to be *positive* if the number of its negative edges is even. Otherwise, it is called *negative*. In particular, the number of positive triangles that contain a fixed vertex i is denoted by T_i^+ .

If $\mathbf{x} = (x_1, x_2, \dots, x_n)^\top$ is an eigenvector associated with the Laplacian eigenvalue μ , then from $L_{\dot{G}}\mathbf{x} = \mu\mathbf{x}$, we get the *eigenvalue equation* (for μ):

$$\mu x_i = d_i x_i - \sum_{u \sim i} \sigma(ui) x_u. \tag{1}$$

Since every graph can be interpreted as a signed graph with all the edges being positive, all notions defined in this section are automatically defined for graphs. Observe that the standard Laplacian matrix of a graph G coincides with the Laplacian matrix of the same graph where it is interpreted as a signed graph. The standard signless Laplacian matrix of G coincides with the Laplacian matrix of the signed graph obtained by reversing the sign of all the edges of G .

3. Results

Our main contribution reads as follows.

Theorem 3.1. *For the Laplacian index μ_1 of a signed graph \dot{G} with n vertices,*

$$\mu_1 \leq \max \left\{ \sqrt{2(d_i^2 + d_i m_i - 2T_i^+)} : 1 \leq i \leq n \right\}, \tag{2}$$

where, for a vertex i , d_i, m_i and T_i^+ denote its degree, average 2-degree and the number of positive triangles containing i , respectively.

Equality holds if and only if \dot{G} is switching equivalent to a regular signed graph with all edges being negative.

Proof. Let \mathbf{x} be an eigenvector associated with μ_1 , and let x_i be the maximum absolute value of the coordinates of \mathbf{x} . Without loss of generality, we may assume that $x_i = 1$. Using the eigenvalue equation (1), we get

$$\mu_1 = d_i - \sum_{u \sim i} \sigma(ui) x_u = d_i - \sum_{\substack{u \overset{+}{\sim} i \\ u \overset{\sim}{\sim} j}} x_u - \sum_{\substack{u \overset{+}{\sim} i \\ u \overset{+}{\sim} j}} x_u - \sum_{\substack{u \overset{+}{\sim} i \\ u \overset{-}{\sim} j}} x_u + \sum_{\substack{u \overset{-}{\sim} i \\ u \overset{\sim}{\sim} j}} x_u + \sum_{\substack{u \overset{-}{\sim} i \\ u \overset{+}{\sim} j}} x_u + \sum_{\substack{u \overset{-}{\sim} i \\ u \overset{-}{\sim} j}} x_u. \tag{3}$$

Similarly, for an arbitrary neighbour of i , say j , we have

$$\mu_1 x_j = d_j x_j - \sum_{u \sim j} \sigma(uj) x_u = d_j x_j - \sum_{\substack{u \overset{+}{\sim} j \\ u \overset{\sim}{\sim} i}} x_u - \sum_{\substack{u \overset{+}{\sim} j \\ u \overset{+}{\sim} i}} x_u - \sum_{\substack{u \overset{+}{\sim} j \\ u \overset{-}{\sim} i}} x_u + \sum_{\substack{u \overset{-}{\sim} j \\ u \overset{\sim}{\sim} i}} x_u + \sum_{\substack{u \overset{-}{\sim} j \\ u \overset{-}{\sim} i}} x_u + \sum_{\substack{u \overset{-}{\sim} j \\ u \overset{+}{\sim} i}} x_u. \tag{4}$$

By adding (4) to (3), we get

$$\mu_1(1 + x_j) = d_i + d_j x_j - \sum_{\substack{u \overset{+}{\sim} i \\ u \overset{\sim}{\sim} j}} x_u - 2 \sum_{\substack{u \overset{+}{\sim} i \\ u \overset{+}{\sim} j}} x_u + \sum_{\substack{u \overset{-}{\sim} i \\ u \overset{\sim}{\sim} j}} x_u + 2 \sum_{\substack{u \overset{-}{\sim} i \\ u \overset{-}{\sim} j}} x_u - \sum_{\substack{u \overset{+}{\sim} j \\ u \overset{\sim}{\sim} i}} x_u + \sum_{\substack{u \overset{-}{\sim} j \\ u \overset{\sim}{\sim} i}} x_u.$$

This implies

$$\begin{aligned} \mu_1(1 + x_j) &\leq d_i + d_j + d_i^+ - c_{ij}^+ + 2c_{ij}^{++} + d_i^- - c_{ij}^- + 2c_{ij}^{--} + d_j^+ - c_{ji}^+ + d_j^- - c_{ji}^- \\ &= 2d_i + 2d_j - 2c_{ij} + 2c_{ij}^{++} + 2c_{ij}^{--} = 2d_i + 2d_j - 2c_{ij}^{+-} - 2c_{ij}^{-+}. \end{aligned} \tag{5}$$

Taking the summation over all j such that $j \sim i$, we get

$$\mu_1 \left(d_i^- + \sum_{j \sim i} x_j \right) \leq 2d_i d_i^- + 2 \sum_{j \sim i} d_j - 2 \sum_{j \sim i} T_{ij}^+. \tag{6}$$

Acting in a similar way, by subtracting (4) from (3), we get

$$\mu_1(1 - x_j) = d_i - d_j x_j - \sum_{\substack{u \overset{+}{\sim} i \\ u \overset{\sim}{\sim} j}} x_u - 2 \sum_{\substack{u \overset{+}{\sim} i \\ u \overset{+}{\sim} j}} x_u + \sum_{\substack{u \overset{-}{\sim} i \\ u \overset{\sim}{\sim} j}} x_u + 2 \sum_{\substack{u \overset{-}{\sim} i \\ u \overset{-}{\sim} j}} x_u + \sum_{\substack{u \overset{+}{\sim} j \\ u \overset{\sim}{\sim} i}} x_u - \sum_{\substack{u \overset{-}{\sim} j \\ u \overset{\sim}{\sim} i}} x_u,$$

which gives

$$\begin{aligned} \mu_1(1 - x_j) &\leq d_i + d_j + d_i^+ - c_{ij}^+ + 2c_{ij}^{+-} + d_i^- - c_{ij}^- + 2c_{ij}^{-+} + d_j^+ - c_{ji}^+ + d_j^- - c_{ji}^- \\ &= 2d_i + 2d_j - 2c_{ij}^+ + 2c_{ij}^{+-} + 2c_{ij}^{-+} = 2d_i + 2d_j - 2c_{ij}^{++} - 2c_{ij}^{--}. \end{aligned} \tag{7}$$

Taking the summation over all j such that $j \overset{\pm}{\sim} i$, we get

$$\mu_1 \left(d_i^+ - \sum_{j \overset{\pm}{\sim} i} x_j \right) \leq 2d_i d_i^+ + 2 \sum_{j \overset{\pm}{\sim} i} d_j - 2 \sum_{j \overset{\pm}{\sim} i} T_{ij}^+. \tag{8}$$

By taking the sum of (6) and (8), we get

$$\mu_1 \left(d_i - \sum_{j \sim i} \sigma(ij)x_j \right) \leq 2d_i^2 + 2 \sum_{j \sim i} d_j - 2 \sum_{j \sim i} T_{ij}^+.$$

By using the eigenvalue equation once again, we arrive at

$$\mu_1^2 \leq 2d_i^2 + 2 \sum_{j \sim i} d_j - 2 \sum_{j \sim i} T_{ij}^+,$$

i.e.,

$$\mu_1 \leq \sqrt{2d_i^2 + 2d_i m_i - 4T_i^+}.$$

Finally, taking the maximum over all vertices, we obtain (2).

If equality in (2) holds, then we have equality in (5) and (7). Considering both, we get

$$\begin{aligned} x_j = 1 &\quad \text{if } i \overset{-}{\sim} j, & \text{and} & \quad x_u = 1 &\quad \text{if } u \overset{-}{\sim} j, \\ x_j = -1 &\quad \text{if } i \overset{+}{\sim} j, & & \quad x_u = -1 &\quad \text{if } u \overset{+}{\sim} j. \end{aligned} \tag{9}$$

In other words, the coordinates of \mathbf{x} are equal in absolute value and if two vertices are adjacent by a negative edge then the corresponding coordinates are equal, while if two vertices are adjacent by a positive edge then the corresponding coordinates differ in sign. Since $x_i = 1$, it follows by (9) that every pair of vertices adjacent by a negative edge correspond to 1 in \mathbf{x} . By making a switch with respect to all the vertices which correspond to -1 in \mathbf{x} , we arrive at the signed graph, say \dot{H} , with all the edges being negative. Simultaneously, \mathbf{x} is transformed to the all-1 eigenvector \mathbf{j} which corresponds to the Laplacian index of \dot{H} . Since $L_{\dot{H}}$ is, in fact, the signless Laplacian matrix of the underlying graph H , we conclude that \dot{H} must be regular (as \mathbf{j} corresponds to its largest eigenvalue – see [6, p. 23]).

Conversely, if \dot{G} is switching equivalent to the regular signed graph (of vertex degree r) with all the edges being negative, then (2) reduces to $\mu_1 \leq \sqrt{4r^2} = 2r$ (since $T_i^+ = 0$, for all i). Moreover, the signless Laplacian index of a regular graph is equal to $2r$, and we are done. □

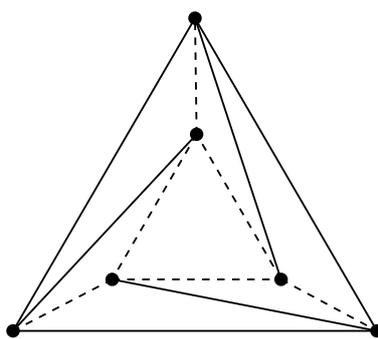


Figure 1: The signed graph for Example 3.1. Negative edges are dashed.

A similar proof technique in case of unsigned graphs can be found in Das’s [1]. Our proof differs since it concerns signed graphs, has a more complicated computation which in particular contains a part that results in inequality (6) and, possibly the most important, resolves the case of equality. Here is an example.

Example 3.1. Consider the signed graph illustrated in Figure 1. Its Laplacian index is equal to 6. Since every vertex belongs to exactly two positive triangles, the upper bound of (2) is equal to $2\sqrt{14} \approx 7.48$. By taking all the edges to be positive, we obtain the signed graph whose Laplacian index is also equal to 6, but now the upper bound gives a better estimate since every vertex belongs to exactly 4 positive triangles; it is $4\sqrt{3} \approx 6.93$. Of course, by taking all the edges to be negative, we get the equality in (2); both sides are equal to 8.

Consider now the particular case of graphs. For the Laplacian index, the bound (2) remains unchanged, since every triangle of a graph is positive.

For the signless Laplacian index κ_1 , the upper bound (2) reduces to

$$\kappa_1 \leq \max \left\{ \sqrt{2d_i(d_i + m_i)} : 1 \leq i \leq n \right\}, \tag{10}$$

since the corresponding graph is interpreted as a signed graph with all edges being negative. The bound (10) is known and can be found in [5, p. 207]. The same bound is derived for the Laplacian index in [2]; evidently, our bound (2) (used for the Laplacian index of a graph) improves this result.

From (1), we get $\kappa x_i = d_i x_i + \sum_{u \sim i} x_u$ (the eigenvalue equation for the signless Laplacian). Acting as in the proof of Theorem 3.1, if $x_i = 1$ is the maximum absolute value of the coordinates of \mathbf{x} and j is a neighbour of i , we get

$$\kappa_1(1 + x_j) = \kappa_1 + d_j x_j + \sum_{u \sim j} x_u,$$

which yields

$$\kappa_1(1 + x_j) \leq \kappa_1 + 2d_j, \tag{11}$$

Taking the summation over all neighbours of i , we arrive at

$$\kappa_1^2 \leq \kappa_1 d_i + 2d_i m_i.$$

Hence,

$$\kappa_1 \leq \frac{d_i + \sqrt{d_i^2 + 8d_i m_i}}{2},$$

along with equality if and only if G is regular, which follows easily by considering the equality in (11). The latter inequality can be found in [5, p. 207], along with a different proof. By [3], for signed graphs with the same underlying graph, the maximum index is attained in the case of that with all edges being negative, up to a switching. In other words, every upper bound for κ_1 of G is simultaneously an upper bound for μ_1 of $\dot{G} = (G, \sigma)$. Thus, we have

$$\mu_1 \leq \frac{d_i + \sqrt{d_i^2 + 8d_i m_i}}{2}. \tag{12}$$

Comparing the bounds (2) and (12), we easily obtain that the former one gives a better estimate whenever

$$T_i^+ \geq \frac{1}{8} d_i (3d_i - \sqrt{d_i(d_i + 8m_i)}).$$

This, in particular, occurs whenever the right-hand side of the last inequality is negative.

We conclude by a lower bound for μ_1 . We say that a signed graph is *net-regular* if the difference between positive and the negative degree is constant on the vertex set. Such a signed graph does not need to be regular, but if it is, then we may write r^+ and r^- for the positive and negative vertex degree, along with $r = r^+ + r^-$. Observe that, in this case, $2r^-$ appears as a Laplacian eigenvalue associated with a constant eigenvector.

Theorem 3.2. *For the Laplacian index μ_1 of a signed graph \dot{G} with n vertices,*

$$\mu_1 \geq \frac{\sum_{i=1}^n d_i (d_i^2 + \sum_{j \sim i} d_j - \sum_{j \not\sim i} d_j)}{\sum_{i=1}^n d_i^2}, \tag{13}$$

where d_i denotes the degree of i .

Equality holds if \dot{G} is switching equivalent to a regular and net-regular signed graph with $2r^-$ in the role of the Laplacian index.

Proof. For $L_{\dot{G}} = (l_{i,j})$ and $\mathbf{d} = (d_1, d_2, \dots, d_n)^\top$, using the Rayleigh principle, we get

$$\mu_1 \geq \frac{\mathbf{d}^\top L_{\dot{G}} \mathbf{d}}{\mathbf{d}^\top \mathbf{d}} = \frac{\sum_{i=1}^n \sum_{j=1}^n d_i l_{i,j} d_j}{\sum_{i=1}^n d_i^2} = \frac{\sum_{i=1}^n (d_i^3 - \sum_{j \sim i} \sigma(i,j) d_i d_j)}{\sum_{i=1}^n d_i^2},$$

which leads to (13).

If \dot{G} is switching equivalent to a regular (of degree r) and net-regular signed graph, then we have

$$\mu_1 \geq \frac{nr(r^2 + r^-r - r^+r)}{nr^2} = r + r^- - r^+ = 2r^-.$$

By the assumption, we have $\mu_1 = 2r^-$, and the result follows. □

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