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Connected non-complete signed graphs which have symmetric spectrum but are not sign-symmetric

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ABSTRACT

A signed graph $\dot{G}$ is called sign-symmetric if it is switching isomorphic to its negation $-\dot{G}$, where $-\dot{G}$ is obtained by reversing the sign of every edge of $\dot{G}$. The authors of Belardo et al. (2018) constructed a complete signed graph that is not sign-symmetric, but has a symmetric spectrum and posted the following problem: Are there connected non-complete signed graphs whose spectrum is symmetric but they are not sign-symmetric? In this paper we positively address this problem. Our examples include infinite families constructed on the basis of the Cartesian product and the corona product of signed graphs. We note that the same problem was first resolved in Ghorbani et al. (2020) by means of different constructions.

1. Introduction

A *signed graph* $\dot{G}$ is a pair (G, σ), where $G = (V, E)$ is a simple unsigned graph, called the *underlying graph*, and $\sigma : E \longrightarrow \{1, -1\}$ is the *sign function* or the *signature*. The number of vertices of $\dot{G}$ is called the *order* and denoted by n. The edge set of $\dot{G}$ is composed of subsets of positive and negative edges. The *adjacency matrix* $A_{\dot{G}}$ of $\dot{G}$ is obtained from the adjacency matrix of its underlying graph by reversing the sign of all 1s which correspond to negative edges. The *eigenvalues* of $\dot{G}$ are identified as the eigenvalues of $A_{\dot{G}}$, and they form the *spectrum* of $A_{\dot{G}}$.

A detailed introduction to the theory of spectra of signed graphs can be found in Zaslavsky's [1]. Accordingly, many concepts in the framework of signed graphs are transferred from the domain of unsigned ones. For example, we say that a signed graph is connected, complete, bipartite or regular if the same holds for its underlying graph. The *degree* of a vertex is equal to its degree in the underlying graph. We say that a cycle in a signed graph is *positive* if it contains an even number of negative edges. Otherwise, it is said to be *negative*. We say that signed graphs $\dot{G}_1$ and $\dot{G}_2$ are *switching isomorphic* if there is a monomial $(0, 1, -1)$-matrix P such that $A_{\dot{G}_2} = P^{-1} A_{\dot{G}_1} P$. Evidently, if signed graphs are switching isomorphic, then there is a bijection (also known as a switching isomorphism) between their sets of vertices which preserves vertex degrees and adjacencies between the vertices. Switching isomorphic signed graphs share the same spectrum. The *negation* of $\dot{G}$, denoted by $-\dot{G}$, is obtained by reversing the sign of every edge of $\dot{G}$. In particular, a signed graph $\dot{G}$ is said to be *sign-symmetric* if it is switching isomorphic to its negation.

We know from [2] that the spectrum of a sign-symmetric signed graph is symmetric (with respect to the origin). Considering the question of existence of signed graphs whose spectrum is symmetric but they are not sign-symmetric, the authors of the same reference have noticed that such signed graphs are necessarily non-bipartite, and then proved that there are complete signed graphs that positively address this question. An example is illustrated in Fig. 1.

We briefly say that a signed graph which has a symmetric spectrum but is not sign-symmetric is an *SSNSS signed graph*. Throughout the paper we exclusively reserve the symbol $\dot{S}$ to correspond to the signed graph of the mentioned figure. We remark that complete SSNSS signed graphs are not easily constructed. For example, $\dot{S}$ is constructed on the basis of a known example of an unsigned graph of order 8 which shares the same Seidel spectrum with its complement, but is not (Seidel-)switching isomorphic to it [3]. We believe that the reader is familiar with the Seidel matrix of a graph and the concept of Seidel switching; in fact, the adjacency matrix of a complete signed graph $\dot{G}$ coincides with the Seidel matrix of a graph induced by negative edges of $\dot{G}$.

The authors of [2] formulated the following problem.

Problem 1.1. Are there connected non-complete SSNSS signed graphs?

In [4], Ghorbani et al. constructed some infinite families of connected non-complete SSNSS signed graphs. Moreover, they proved the existence of such signed graphs for every order ≥ 6. In this paper we construct a sequence of examples that positively address the same problem. Our examples differ from those of [4] and include infinite families of SSNSS signed graphs.

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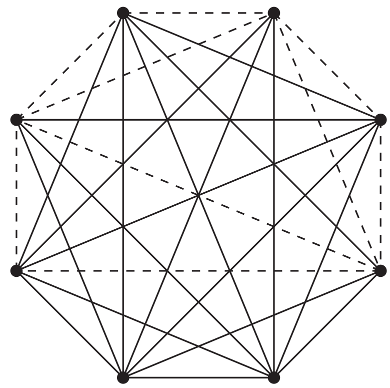

Fig. 1. The connected complete SSNSS signed graph $\dot{S}$. In this and forthcoming figures, negative edges are dashed.

2. Connected non-complete SSNSS signed graphs

Our notation is standard. For example, we use K_n and P_n to denote the unsigned complete graph and the unsigned path with n vertices, respectively. If the signed graphs $\dot{G}_1, \dot{G}_2$ are switching isomorphic, then we write $\dot{G}_1 \cong \dot{G}_2$.

Let $\dot{G}_1$ and $\dot{G}_2$ be signed graphs with vertex sets $\{u_1, u_2, \dots, u_{n_1}\}$ and $\{v_1, v_2, \dots, v_{n_2}\}$, respectively. The set of vertices of the *Cartesian product* $\dot{G}_1 \square \dot{G}_2$ is the Cartesian product of the sets of vertices of $\dot{G}_1$ and $\dot{G}_2$, and the vertices (u_i, v_j) and (u_k, v_l) are joined by a positive (resp. negative) edge if and only if $u_i = u_k$ and v_j, v_l are joined by a positive (negative) edge or u_i, u_k are joined by a positive (negative) edge and $v_j = v_l$. Observe that $\dot{G}_1 \square \dot{G}_2$ consists of n_2 copies of $\dot{G}_1$ and additional edges which occur between a vertex in the jth copy and the same vertex in the lth copy precisely if v_j and v_l are adjacent in $\dot{G}_2$; the sign of such an edge is transferred from $\dot{G}_2$. Clearly, $\dot{G}_1 \square \dot{G}_2$ is connected if and only if $\dot{G}_1$ and $\dot{G}_2$ are connected.

If the eigenvalues of $\dot{G}_1$ are $\lambda_1, \lambda_2, \dots, \lambda_{n_1}$ and the eigenvalues of $\dot{G}_2$ are $\mu_1, \mu_2, \dots, \mu_{n_2}$, then the eigenvalues of $\dot{G}_1 \square \dot{G}_2$ are $\lambda_i + \mu_j$, for $1 \leq i \leq n_1$, $1 \leq j \leq n_2$, see [5].

We start with the following lemma.

Lemma 2.1. *If spectra of signed graphs $\dot{G}_1$ and $\dot{G}_2$ are symmetric, then the spectrum of $\dot{G}_1 \square \dot{G}_2$ is also symmetric.*

Proof. Assume that the spectrum of $\dot{G}_1 \square \dot{G}_2$ is not symmetric. Then there is a non-zero eigenvalue of the form $\lambda_i + \mu_j$ (for fixed eigenvalues λ_i of $\dot{G}_1$ and μ_j of $\dot{G}_2$), such that its negation $-\lambda_i - \mu_j$ is not an eigenvalues of the Cartesian product. But since spectra of $\dot{G}_1$ and $\dot{G}_2$ are symmetric, $-\lambda_i$ is an eigenvalue of $\dot{G}_1$ and $-\mu_j$ is an eigenvalue of $\dot{G}_2$, which means that their sum belongs to the spectrum of $\dot{G}_1 \square \dot{G}_2$ and contradicts the assumption. □

Here is our first construction of SSNSS signed graphs.

Theorem 2.2. *Let $\dot{G}_1$ be a complete SSNSS signed graph and $\dot{G}_2$ be a connected signed graph with a symmetric spectrum, such that the underlying graph G_1 is not an induced subgraph of the underlying graph G_2. Then $\dot{G}_1 \square \dot{G}_2$ is a connected SSNSS signed graph.*

Proof. First, $\dot{G}_1 \square \dot{G}_2$ is connected (since $\dot{G}_1$ and $\dot{G}_2$ are connected). Next, from Lemma 2.1, we immediately get that the spectrum of $\dot{G}_1 \square \dot{G}_2$ is symmetric, and so it remains to show that $\dot{G}_1 \square \dot{G}_2$ is not switching isomorphic to its negation.

In an intermediate step we prove that if n_1, n_2 are the orders of G_1, G_2 then a maximal clique (i.e., maximal complete induced subgraph) of the underlying graph $G_1 \square G_2$ has n_1 vertices and there are exactly n_2 such cliques isomorphic to the n_2 disjoint copies of G_1. First, by definition of the Cartesian product, $G_1 \square G_2$ contains the n_2 disjoint copies of G_1. Assume that K is either a larger clique or a clique with n_1 vertices but distinct from the previous ones. In both cases it contains a pair of vertices $(u_i, v_j), (u_k, v_l)$ where i, k are fixed distinct integers from 1 to n_1; for otherwise, K would be one of the n_2 existing cliques. If so, then there must be $j = l$; for otherwise, the mentioned vertices would not be adjacent. Moreover, K cannot contain a vertex whose second coordinate differs from v_j, since such a vertex is not adjacent to both $(u_i, v_j), (u_k, v_j)$. In other words, K is an induced subgraph of G_2. Since G_2 does not contain G_1, we conclude that K has less than n_1 vertices, which contradicts the initial assumption.

Assume now by way of contradiction that $\dot{G}_1 \square \dot{G}_2$ is switching isomorphic to its negation. By the previous part of the proof, a corresponding switching isomorphism must map $\dot{G}_1$ onto a copy of $-\dot{G}_1$ in $-\dot{G}_1 \square \dot{G}_2$, but this is impossible since $\dot{G}_1$ is not switching isomorphic to its negation, a contradiction. □

It follows by definition that the Cartesian product is complete if and only if one of $\dot{G}_1, \dot{G}_2$ consists of a single vertex, while the other is complete. Therefore, apart from this simple situation, the previous theorem necessarily produces non-complete SSNSS signed graphs. We proceed with an example.

Example 2.3. The signed graph $\dot{S} \square P_3$ is illustrated in Fig. 2. It is a connected non-complete SSNSS signed graph, by Theorem 2.2. In this example, the second signed graph of the product (so, P_3) is bipartite, but non-bipartite ones with symmetric spectrum can also be taken into account. Say, such a signed graph can be obtained by taking two triangles, one positive the other negative, and inserting a path between a pair of their vertices.

We now give an infinite family of connected non-complete SSNSS signed graphs.

Theorem 2.4. *For $i \geq 0$, the signed graph $\dot{G}_i$ obtained by the following iterative procedure*

$$\begin{cases} \dot{G}_0 \cong \dot{S}, \\ \dot{G}_{i+1} \cong \dot{G}_i \square K_2, \end{cases}$$

is a connected SSNSS signed graph. It is non-complete for $i \geq 1$.

Proof. $\dot{G}_i$ is connected since it is the Cartesian product of connected signed graphs.

We prove that $\dot{G}_i$ is an SSNSS signed graph by induction whose basis is easily formed by taking into account that $\dot{S}$ is an SSNSS signed graph. Assume that the claim holds for $\dot{G}_i$ and consider $\dot{G}_{i+1} \cong \dot{G}_i \square K_2$. By definition of the Cartesian product, every complete induced subgraph of $\dot{G}_{i+1}$ with 8 vertices is isomorphic to $\dot{S}$ and every complete induced subgraph of $-\dot{G}_{i+1}$ with 8 vertices is isomorphic to $-\dot{S}$. Therefore, a switching isomorphism between $\dot{G}_{i+1}$ and $-\dot{G}_{i+1}$ necessarily maps $\dot{S}$ onto $-\dot{S}$, which is impossible as they are not switching isomorphic.

Evidently, for $i \geq 1$, $\dot{G}_i$ is non-complete by the argument mentioned below Theorem 2.2. □

The signed graph $\dot{G}_1$ of the previous theorem is obtained by deleting the third copy of $\dot{S}$ from Fig. 2.

We proceed now with a construction on the basis of another product. The *corona product* (briefly, *corona*) $\dot{G}_1 \circ \dot{G}_2$ of signed graphs $\dot{G}_1$ and $\dot{G}_2$ is obtained by taking $\dot{G}_1$ (which has n_1 vertices) and $n_1 \dot{G}_2$ (n_1 copies of $\dot{G}_2$), and then inserting a positive edge between the ith vertex of $\dot{G}_1$ and every vertex in the ith copy of $\dot{G}_2$, for $1 \leq i \leq n_1$. In what follows we consider the particular case in which $\dot{G}_2 \cong K_1$. We first compute the spectrum of such a corona.

Lemma 2.5. *If $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of $\dot{G}$, then all the eigenvalues of $\dot{G} \circ K_1$ are determined by the equations $x^2 - \lambda_i x - 1 = 0$, for $1 \leq i \leq n$.*

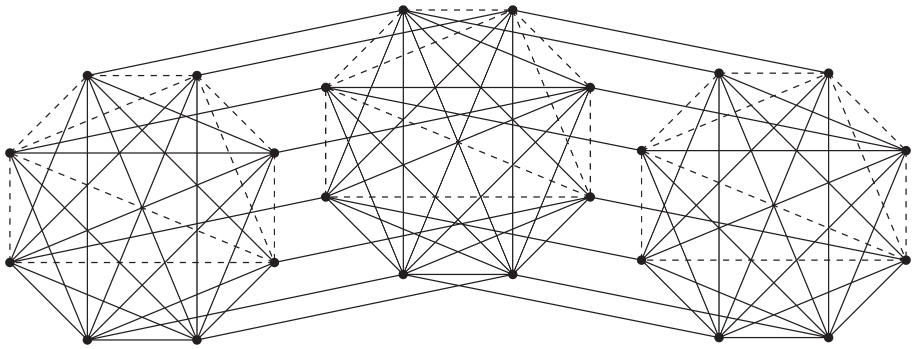

Fig. 2. The connected non-complete SSNSS signed graph $\dot{S}\square P_3$.

Proof. The adjacency matrix of $\dot{G}\circ K_1$ is given by

$$A_{\dot{G}\circ K_1} = \begin{pmatrix} A_{\dot{G}} & I_n \\ I_n & O_n \end{pmatrix},$$

where I_n and O_n are the $n \times n$ identity and all-0 matrix, respectively. Since the blocks of $xI_{2n} - A_{\dot{G}\circ K_1}$ all commute with each other, its determinant is computed as the 2×2 determinant, which leads to

$$\det\left(xI_{2n} - A_{\dot{G}\circ K_1}\right) = \det\left((xI_n - A_{\dot{G}}) \cdot xI_n - I_n^2\right) = \begin{cases} x^n \det\left(\frac{x^2-1}{x} I_n - A_{\dot{G}}\right), & x \neq 0, \\ -\det(I_n), & x = 0. \end{cases}$$

Therefore x is an eigenvalue of $A_{\dot{G}\circ K_1}$ if and only if $\frac{x^2-1}{x} = \lambda_i$, for $1 \leq i \leq n$, and we are done. □

We now determine whether the spectrum of $\dot{G}\circ K_1$ is symmetric.

Lemma 2.6. *If the spectrum of $\dot{G}$ is symmetric, then the spectrum of $\dot{G}\circ K_1$ is also symmetric.*

Proof. By Lemma 2.5, if λ is an eigenvalue of $\dot{G}$, then $\frac{1}{2}(\lambda \pm \sqrt{4+\lambda^2})$ are the eigenvalues of $\dot{G}\circ K_1$. They are symmetric to the eigenvalues $\frac{1}{2}(-\lambda \mp \sqrt{4+\lambda^2})$ that arise from the eigenvalue $-\lambda$ of $\dot{G}$, and we are done. □

We are ready to present another iterative construction.

Theorem 2.7. *For $i \geq 0$, the signed graph $\dot{G}_i$ obtained by the following iterative procedure*

$$\begin{cases} \dot{G}_0 \cong \dot{S}, \\ \dot{G}_{i+1} \cong \dot{G}_i\circ K_1, \end{cases}$$

is a connected SSNSS signed graph. It is non-complete for $i \geq 1$.

Proof. G_i is connected and non-complete for $i \geq 1$ by definition of corona, while its spectrum is symmetric by Lemma 2.6. Observe that, for every i, G_i contains exactly one complete subgraph with 8 vertices, i.e., $\dot{S}$, and therefore it is not isomorphic to its negation by the argument exploited in the proof of Theorem 2.4. □

The signed graph $\dot{G}_1$ of the previous theorem is illustrated in Fig. 3.

Remark 2.8. Our motivation was to obtain some examples which address Problem 1.1, but not to give a large number of complicated constructions of SSNSS signed graphs. In relation to this, an observant reader will notice that the constructions of Theorems 2.4 and 2.7 can be generalized, say by considering an arbitrary SSNSS signed graph instead of $\dot{S}$.

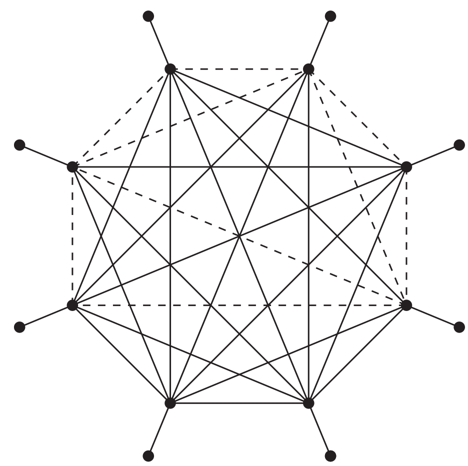

Fig. 3. The connected non-complete SSNSS signed graph $\dot{S}\circ K_1$.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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