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NESTED GRAPHS WITH BOUNDED SECOND LARGEST (SIGNLESS LAPLACIAN) EIGENVALUE*

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Abstract. Nested split and double nested graphs (commonly named nested graphs) are considered. General statements regarding the signless Laplacian spectra are proven, and the nested graphs whose second largest signless Laplacian eigenvalue is bounded by a fixed integral constant are studied. Some sufficient conditions are provided and a procedure for classifying such graphs in particular cases is provided. Some connections between their structure and some (not only the second) eigenvalues of their signless Laplacians are developed. All double nested graphs whose second largest eigenvalue does not exceed $\sqrt{2}$ are determined.

Key words. Nested split graphs, double nested graphs, (signless Laplacian) spectrum, second largest eigenvalue.

AMS subject classifications. 05C50.

1. Introduction. Let G be a graph on n vertices with adjacency matrix $A = A_G$. The characteristic polynomial $P_G(x) = \det(xI - A)$ of A is called the *characteristic polynomial* of G. The matrix $Q = D + A$, where D is the diagonal matrix of vertex-degrees in G, is called the signless Laplacian matrix of G, and $Q_G(x) = \det(xI - Q)$ is the *Q-polynomial* of G. The eigenvalues and the spectrum of A (resp. Q) are also called the *eigenvalues* (resp. *signless Laplacian eigenvalues*; briefly *Q-eigenvalues*) and the *spectrum* (resp. *signless Laplacian spectrum*; briefly *Q-spectrum*) of G. Since the mentioned matrices are real and symmetric, their eigenvalues are real. Thus, the spectrum and the signless Laplacian spectrum we shall denote by $\lambda_1(G), \lambda_2(G), \ldots, \lambda_n(G)$, and $\kappa_1(G), \kappa_2(G), \ldots, \kappa_n(G)$, respectively. In the sequel we shall usually suppress G in our notation; in addition, we assume that $\lambda_i \geq \lambda_{i+1}$ and $\kappa_i \geq \kappa_{i+1}$, $i = 1, 2, \ldots, n-1$.

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The largest eigenvalues in these spectra will be called the *index* and the *Q-index*, respectively.

By the term *nested graphs* we refer to two classes of graphs: the *nested split graphs* (briefly, NSGs) and their bipartite equivalents *double nested graphs* (briefly, DNGs). We recall their definitions in the next section. Both classes play an important role in the research concerning the graphs with maximal (Q-)index. Namely, it is known that graph with maximal index or maximal Q-index and fixed order and size is an NSG (see, for example, [5, p. 231]) or a DNG if it is bipartite (see [1, 3]).

The problem of determining the graphs whose second largest eigenvalue is bounded by some (relatively small) number is well studied. The graphs whose second largest eigenvalue does not exceed $\frac{1}{3}$ or $\sqrt{2}-1$ are determined, while the graphs satisfying $\lambda_2 \leq \frac{\sqrt{5}-1}{2}$ are well characterized but not completely determined (see [9]). Additionally, there are various results regarding the cases $\lambda_2 \leq 1$ (see [11] and the references therein), $\lambda_2 \leq \sqrt{2}$ (see [12]) and $\lambda_2 \leq 2$ (see [9]), but they are still unsolved. Such research mostly focusses on the classification of graphs or the description of their structure regarding their spectral properties (especially, regarding their second largest eigenvalue). So far there are no such results concerning the second largest Q-eigenvalue (see [2, 7] for example).

The graphs whose second largest eigenvalue does not exceed 1 (and further $\sqrt{2}$ or 2) are determined only if they belong to some specific classes (not to be listed here). Even then, the corresponding bound is a relatively small number. For example, all NSGs satisfying $\lambda_2 \leq 1$ are determined in [10] and [8], and here we determine all DNGs satisfying $\lambda_2 \leq \sqrt{2}$. It turns out that the graphs belonging to the same classes can be much easily sorted according to their second largest Q-eigenvalues. Moreover, it turns out that some structural properties of these graphs are closely connected to their second largest (but also some other) Q-eigenvalues.

The paper is organized as follows. In Section 2 some preliminary definitions and results are given in order to make the paper more self-contained. In Section 3 we give some general results regarding the signless Laplacian spectrum. Next we consider the nested graphs whose second largest Q-eigenvalue does not exceed a prescribed integral constant; we provide some sufficient conditions for this property and consider some particular cases. Some structural properties of these graphs are also given. In Section 4 we determine all DNGs whose second largest eigenvalue does not exceed $\sqrt{2}$. The graphs obtained are given in the Appendix.

2. Preliminaries. The graphs having no induced subgraphs $2K_2$, P_4 or C_4 are called (by P. Hansen) *nested split graphs* (or NSGs). The vertices of an arbitrary NSG can be partitioned into $2h$ cells $\bigcup_{i=1}^{h} U_i$ and $\bigcup_{i=1}^{h} V_i$, where the subgraph induced by

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$\bigcup_{i=1}^{h} U_i$ (resp. $\bigcup_{i=1}^{h} V_i$) is a complete (resp. totally disconnected) graph, while all vertices in U_i are adjacent to all vertices in V_j if and only if $i \leq j$.

Similarly the vertex set of any connected *double nested graph* (or DNG) consists of two colour classes (or co-cliques), and both of which are partitioned into h non-empty cells $\bigcup_{i=1}^{h} U_i$ and $\bigcup_{i=1}^{h} V_i$, respectively. All vertices in U_i are adjacent to all vertices in $\bigcup_{j=1}^{h+1-i} V_j$, for $i = 1, 2, \ldots, h$.

We use the common name, *nested graphs*, for both NSGs and DNGs. Let (in both cases) $m_i = |U_i|, n_i = |V_i|,\ i = 1, \ldots, h$. Then we have that the set of all vertices of the corresponding nested graph G is $V = \bigcup_{i=1}^{h} U_i \cup \bigcup_{i=1}^{h} V_i$, while $\nu = |V| = \sum_{i=1}^{h}(m_i + n_i)$. An arbitrary NSG (resp. DNG) will be denoted by

$$\mathrm{NSG}(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h) \text{ (resp. } \mathrm{DNG}(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h)).$$

In general, an arbitrary nested graph will be denoted by $NG(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h)$. Note that an NSG (resp. a DNG) is connected whenever m_1 (resp. both m_1 and n_1) is greater than zero. If any of the remaining parameters is equal to zero, we again get a nested graph with a smaller parameter h, so we usually assume that each of these parameters is greater than zero.

Let us now introduce the so-called *divisor concept*, which will be widely used in this paper. Given an $s \times s$ matrix $D = (d_{ij})$, let the vertex set of a multigraph G be partitioned into non-empty subsets $V_1, V_2, \ldots, V_s$ so that for any $i, j \in \{1, 2, \ldots, s\}$ each vertex from V_i is adjacent to exactly d_{ij} vertices of V_j. The multigraph H with adjacency matrix D is called a *front divisor* of G, or briefly, a *divisor* of G ([6, Definition 2.4.4]). (Note that the Q-matrix of any graph can be considered as the adjacency matrix of the corresponding multigraph bearing in mind that each diagonal entry is equal to the number of loops of the corresponding vertex, and therefore the previous concept can be applied, as well.)

A (Q-)eigenvalue of a graph G is a *main (Q-)eigenvalue* provided the corresponding (Q-)eigenvector is not orthogonal to $(1, 1, \ldots, 1)^T$ (compare [6, p. 25, Theorem 2.2.3]). Otherwise, the (Q-)eigenvalue is called a *non-main (Q-)eigenvalue*. The *main part* of the (Q-)spectrum of G contains only its main (Q-)eigenvalues.

The characteristic polynomial of a divisor divides the characteristic polynomial (or a Q-polynomial) of a graph (cf. [6, p. 38]), and due to [6, Theorem 2.4.5] of the (Q-)spectrum of any divisor H of graph G includes the main part of the (Q-)spectrum of G.

If G is an arbitrary graph and u a vertex, then $\Gamma(u)$ and $\Gamma[u]$ denote open and closed neighbourhoods of u, respectively; so $\Gamma(u) = \{v \in V(G) \mid v \sim u\}$ while $\Gamma[u] = \Gamma(u) \cup \{u\}$. Two vertices are *duplicate* (*coduplicate*) if their open (resp. closed) neighbourhoods are the same. It is known that any pair of duplicate (resp. codu-

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plicate) vertices gives rise to an eigenvector of G for 0 (resp. -1) defined as follows: all its entries are zero except those corresponding to u and v which can be taken to be 1 and -1, or vice versa. Thus any collection with k mutually duplicate (resp. coduplicate) vertices gives rise to $k-1$ linearly independent eigenvectors for 0 (resp. -1). Similarly, any collection of k mutually duplicate (resp. coduplicate) vertices of degree d in a graph G gives $k-1$ Q-eigenvalues of G all equal to d (resp. $d-1$) where the corresponding Q-eigenvectors are formed in the same way. In addition, all these eigenvalues and Q-eigenvalues are non-main (according to the corresponding definition). We finish this section with the following formula (compare [4, Theorem 2.17]):

$$P_{S(G)}(x) = x^{m-n} Q_G(x^2), \tag{2.1}$$

where G has n vertices, m edges, while $S(G)$ denotes its subdivision (the graph obtained by inserting a vertex in each of its edges).

3. Nested graphs with bounded second largest Q-eigenvalue.

The complete product $G_1 \nabla G_2$ of (disjoint) graphs G_1 and G_2 is the graph obtained from the union of disjoint copies of the graphs G_1 and G_2 by joining each vertex of G_1 to each vertex of G_2. The following result concerns the Q-polynomial of the complete product of two regular graphs.

THEOREM 3.1. *Given regular graphs G_1 (on n_1 vertices) and G_2 (on n_2 vertices) having degrees r_1 and r_2, respectively. Then*

$$Q_{G_1 \nabla G_2}(x) = \frac{Q_{G_1}(x-n_2) Q_{G_2}(x-n_1)}{(x-2r_1-n_2)(x-2r_2-n_1)} \big((x-2r_1-n_2)(x-2r_2-n_1) - n_1 n_2\big).$$

Proof. Recall that a Q-eigenvalue of any regular graph G is main if and only if it is equal to its Q-index (compare [4, p. 403], and have in mind that $P_G(x) = Q_G(x+r)$ for regular graphs of the degree r).

We have

$$Q_{G_1 \nabla G_2} = \begin{bmatrix} \mathbf{Q_{G_1} + n_2 I_{n_1}} & \mathbf{J} \\ \mathbf{J^T} & \mathbf{Q_{G_2} + n_1 I_{n_2}} \end{bmatrix},$$

where I_{n_i} $i = 1, 2$ is the unit matrix of the corresponding size, while J denotes $n_1 \times n_2$ matrix with each entry equal to 1. Since the divisor of (the Q-matrix of) $G_1 \nabla G_2$ has the form

$$\begin{bmatrix} 2r_1 + n_2 & n_2 \\ n_1 & 2r_2 + n_1 \end{bmatrix}$$

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we get that its Q-polynomial contains $\big((x-2r_1-n_2)(x-2r_2-n_1)-n_1n_2\big)$ as a factor.

Assume first that both G_1 and G_2 are connected. Let $\kappa_2(G_1),\ldots,\kappa_{n_1}(G_1)$ and $\kappa_2(G_2),\ldots,\kappa_{n_2}(G_2)$ be their non-main Q-eigenvalues, and let $e_1(G_1),\ldots,e_{n_1}(G_1)$ and $e_1(G_2),\ldots,e_{n_2}(G_2)$ be the corresponding Q-eigenvectors. Having in mind that each of these Q-eigenvectors is orthogonal to the "all ones" vector, by direct computation we get that $\kappa_2(G_1)+n_2,\ldots,\kappa_{n_1}(G_1)+n_2$ and $\kappa_2(G_2)+n_1,\ldots,\kappa_{n_2}(G_2)+n_1$ are the Q-eigenvalues of $G_1\nabla G_2$ where $\kappa_i(G_1)+n_2$ (resp. $\kappa_i(G_2)+n_1$) corresponds to the Q-eigenvector whose first n_1 (resp. last n_2) coordinates coincide with the coordinates of $e_i(G_1)$ (resp. $e_i(G_2)$) while the remaining coordinates are zeros. Therefore, we get the above formula.

Now let G_1 and G_2 be disconnected graphs having k and l components, respectively. Then, each non-main Q-eigenvalue of G_1 and G_2 gives the corresponding Q-eigenvalue of $G_1\nabla G_2$ in the same way as above. Since the sum of all Q-eigenvalues of $G_1\nabla G_2$ is equal to the trace of $Q_{G_1\nabla G_2}$, it can be verified that $k-1$ (resp $l-1$) of the remaining Q-eigenvalues are equal to $2r_1-n_2$ (resp. $2r_2-n_1$). ☐

By (2.1) and Theorem 3.1 we have the following result.

Corollary 3.2. *Given regular graphs G_1 (on n_1 vertices) and G_2 (on n_2 vertices) having degrees r_1 and r_2, respectively. Then*

$$P_{S(G_1\nabla G_2)}(x)=x^{\frac{n_1(r_1-2)+n_2(r_2-2)}{2}+n_1n_2}Q_{G_1\nabla G_2}(x^2).$$

We now use Theorem 3.1 to compute the Q-polynomials of two specific kinds of graphs; namely the complete bipartite graphs K_{n_1,n_2} and the graphs $K_{n_1}\nabla n_2K_1$ obtained from $K_{n_1+n_2}$ by removing a clique n_2-vertices. Both polynomials will be used later on.

Corollary 3.3.

$$Q_{K_{n_1,n_2}}(x)=(x-n_1-n_2)(x-n_1)^{n_2-1}(x-n_2)^{n_1-1}x; \tag{3.1}$$

$$Q_{K_{n_1}\nabla n_2K_1}(x)=(x-n_1)^{n_2-1}(x-n_1-n_2+2)^{n_1-1}\times \left(x^2-(3n_1+n_2-2)x+2n_1(n_1-1)\right). \tag{3.2}$$

Now we prove some results regarding nested graphs. It is not hard to check that an arbitrary disconnected NSG or DNG contains at most one non-trivial component and a set of isolated vertices. Therefore, we restrict ourselves to the connected graphs, while with slight modifications all the results can be extended to disconnected cases.

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Lemma 3.4. *Let*

$$G = NG(m_1, \ldots, m_j, m_{j+1}, \ldots, m_h; n_1, \ldots, n_k, n_{k+1}, \ldots, n_h)$$

be a connected nested graph having ν *vertices, and let*

$$G' = NG(m_1, \ldots, m_j + 1, m_{j+1} - 1, \ldots, m_h; n_1, \ldots, n_k, n_{k+1}, \ldots, n_h)$$

and

$$G'' = NG(m_1, \ldots, m_j, m_{j+1}, \ldots, m_h; n_1, \ldots, n_k + 1, n_{k+1} - 1, \ldots, n_h).$$

Then

$$\begin{aligned} &\kappa_i(G) \le \kappa_i(G')\,, \ \ \textit{for } i = 1, \ldots, \nu; \\ &\kappa_i(G) \le \kappa_i(G'')\,, \ \ \textit{for } i = 1, \ldots, \nu, \ \textit{whenever } G \textit{ is a DNG}; \\ &\kappa_i(G) \ge \kappa_i(G'')\,, \ \ \textit{for } i = 1, \ldots, \nu, \ \textit{whenever } G \textit{ is a NSG.} \end{aligned}$$

Proof. Practically, G' is obtained by adding an appropriate number of edges to a single vertex of G, while G'' is obtained in the same way whenever G is a DNG or by removing the appropriate edges if it is an NSG. The result follows from the fact that adding the edges to any graph implies the increasing (not necessarily strict) of all its Q-eigenvalues (see [7]). ◻

Lemma 3.5. *Let* $G = NG(m_1, \ldots, m_h; n_1, \ldots, n_h)$ *be a connected nested graph then*

$$\begin{aligned} &\kappa_2(G) \le \max\left\{\sum_{i=1}^{h} m_i, \sum_{i=1}^{h} n_i\right\}, \ \textit{whenever } G \textit{ is a DNG; and} \\ &\kappa_2(G) \le \nu - 2\,, \ \textit{whenever } G \textit{ is a NSG.} \end{aligned}$$

Proof. Using Lemma 3.4, (3.1), and (3.2) we get

$$\begin{aligned} \kappa_2(\mathrm{DNG}(m_1, \ldots, m_h; n_1, \ldots, n_h)) &\le \kappa_2\left(\mathrm{DNG}\left(\sum_{i=1}^{h} m_i, \sum_{i=1}^{h} n_i\right)\right) \\ &= \kappa_2\left(K_{\sum_{i=1}^{h} m_i, \sum_{i=1}^{h} n_i}\right) \\ &\le \max\left\{\sum_{i=1}^{h} m_i, \sum_{i=1}^{h} n_i\right\}. \end{aligned}$$

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Similarly,

$$\begin{aligned}\kappa_2(\mathrm{NSG}(m_1,\dots,m_h;n_1,\dots,n_h)) &\le \kappa_2\left(\mathrm{NSG}\left(\sum_{i=1}^h m_i, \sum_{i=1}^h n_i\right)\right)\\ &= \kappa_2\left(K_{\sum_{i=1}^h m_i} \nabla \sum_{i=1}^h n_i K_1\right)\\ &= \sum_{i=1}^h (m_i+n_i) - 2\\ &= \nu - 2. \ \square\end{aligned}$$

Note that the corresponding bound for NSGs is given in [13] for the graphs obtained by deleting at most $\nu - 2$ edges from K_ν. An arbitrary NSG is obtained in the same way but the number of the deleted edges can be even larger. The results of Lemma 3.5 can also be compared to [2, Theorem 3.1 and Corollary 3.7].

Before we give a consequence of Lemma 3.5, we take into consideration the remaining Q-eigenvalues. Using the concept explained in the previous section we determine the divisors of both types of nested graphs. It is easy to check that the divisor of a connected $NG(m_1,\dots,m_h;n_1,\dots,n_h)$ has $2h$ Q-eigenvalues (or possibly $2h-1$ if $n_h = 0$ for NSG). The remainder of the Q-spectrum consists of non-main Q-eigenvalues. They are determined in the next theorem.

THEOREM 3.6. *Let $G = NG(m_1,\dots,m_h;n_1,\dots,n_h)$ be a connected nested graph. Then $2h$ of its Q-eigenvalues are determined by its divisor, and the remaining Q-eigenvalues are*

$$\sum_{j=1}^{h+1-i} m_j \quad \textit{with multiplicity } m_i - 1 \ \ (i=1,\dots,h) \quad \textit{and}$$
$$\sum_{j=1}^{h+1-i} n_j \quad \textit{with multiplicity } n_i - 1 \ \ (i=1,\dots,h)$$

if G is a DNG, or

$$\sum_{j=1}^{h} m_j + \sum_{j\ge i} n_j - 2 \quad \textit{with multiplicity } m_i - 1 \ \ (i=1,\dots,h) \quad \textit{and}$$
$$\sum_{j\ge i} m_j \quad \textit{with multiplicity } \ n_i - 1 \ \ (i=1,\dots,h)$$

if G is an NSG.

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Proof. Assume that G is a DNG, then each set U_i or V_i, $i = 1, \ldots, h$, contains mutually duplicate vertices (see the previous section). These vertices give the listed Q-eigenvalues.

If G is an NSG then each set U_i (resp. V_i) contains mutually coduplicate (resp. duplicate) vertices giving the listed Q-eigenvalues. ◻

The next theorem is an immediate consequence of Lemma 3.5.

THEOREM 3.7. *Let G be a connected nested graph with maximum degree $\Delta(G)$. Then $\kappa_2(G) \leq \Delta(G)$ if G is a DNG, and $\kappa_2(G) \leq \Delta(G) - 1$ if G is an NSG.*

Proof. Note that if G is a DNG then $\Delta(G) = \max\left\{\sum_{i=1}^{h} m_i, \sum_{i=1}^{h} n_i\right\}$, while if G is an NSG then $\Delta(G) = \nu - 1$. Since $\kappa_2(G)$ does not exceed the bounds given in Lemma 3.5, we obtain the result. ◻

In fact, the previous theorem provides sufficient conditions that the second largest Q-eigenvalue of a nested graph does not exceed α ($\in \mathbb{N}$) (it can be checked that the given bounds do not hold for any graph). In the following example we determine all NSGs satisfying $\kappa_2 \leq 4$, while the similar procedure can be applied in any other case[1].

EXAMPLE 3.8. *Let $G = NSG(m_1, \ldots, m_h; n_1, \ldots, n_h)$ be connected. If $\kappa_2(G) \leq 4$, then either the maximum vertex degree does not exceed 5 or G is an induced subgraph of one of the following NSGs: $NSG(1, 3; 3)$, $NSG(1, 1, 1; 2, 2)$, $NSG(1, 2; 3, 1)$, $NSG(1, 1, 1; *, 1)$, $NSG(1, 1; *, 2)$, $NSG(1, 1; 2, 3)$.*

Namely, due to Theorem 3.7 we have $\kappa_2(G) \leq 4$ whenever $\Delta(G) \leq 5$. Now it remains to consider the NSGs whose maximum vertex degree is greater than 5. In particular, this means that each vertex in U_1 has degree greater than 5, but using Theorem 3.6 we get that the parameter m_1 must be equal to 1 (otherwise $\kappa_1 > 4$). Having in mind that the remaining Q-eigenvalues given in the same theorem must also be bounded by 4, we obtain very restrictive conditions for the remaining parameters. Finally, by direct computation we get the listed NSGs.

Now we give some relations between the structural properties of nested graphs and their Q-eigenvalues. If, with no loss of generality, we assume that $\sum_{i=1}^{h} n_i = \max\left\{\sum_{i=1}^{h} m_i, \sum_{i=1}^{h} n_i\right\}$ for some DNG G, then $\kappa_2(G) = \Delta(G)$ whenever $m_1 \geq 2$ (see Theorem 3.6). Moreover, in this case we have $\kappa_2(G) = \kappa_3(G) = \cdots = \kappa_{m_1}(G) = \Delta(G)$. Similarly, if G is a connected NSG then $\kappa_2(G) = \Delta(G) - 1$ whenever $m_1 \geq 2$ as well as $\kappa_2(G) = \kappa_3(G) = \cdots = \kappa_{m_1}(G) = \Delta(G)$.

The remaining non-main Q-eigenvalues considered in Theorem 3.6 are all integral and closely connected to the values m_i and n_i ($i = 1, \ldots, h$), describing the graph

[1] Here, in Example 4.11 and in Tables 1-4 (given in Appendix), '$*$' stands for any positive integer.

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structure as well.

As we pointed out, the graph with maximal Q-index of fixed order and size is an NSG. Here we provide the following result.

THEOREM 3.9. *Let $G = NSG(m_1, \ldots, m_h; n_1, \ldots, n_h)$ have $\nu > 2h$ vertices, and let $m_i, n_i > 0, i = 1, \ldots, h$. Then each of non-main Q-eigenvalues mentioned in Theorem* 3.6 *does not exceed $\nu - 2$. All of them attain this bound if $m_1 = \nu - 2h + 1$.*

Proof. Since $\sum_{i=1}^{h}(m_i + n_i) = \nu$ we get that these Q-eigenvalues do not exceed $\nu - 2$, attaining this bound if m_1 has the maximum possible value, i.e., $m_1 = \nu - 2h + 1$, while the remaining parameters are all equal to 1. □

Using (2.1) and [4, Theorem 2.19], the results of this section can be transferred to the adjacency spectrum of subdivision graphs and line graphs.

4. Double nested graphs with $\lambda_2 \leq \sqrt{2}$. In this section we consider the adjacency spectra of specified graphs. In fact, we determine all connected DNGs whose second largest eigenvalue does not exceed $\sqrt{2}$. Since each disconnected DNG contains at most one non-trivial component and a set of isolated vertices, in this way we determine all DNGs satisfying this condition. We start with the results considering the structure and the spectral properties of these graphs. The results starting from Theorem 4.5 are more technical and very similar to the results obtained in [10] and [8], so we only present the complete proof of this theorem. The remaining statements are proved in a very similar way. Additionally, all DNGs obtained are listed in the Appendix. We also give an example concerning DNGs with $\lambda_2 \leq 1$.

To simplify some expressions we write $(a_1, a_2, \ldots, a_i^{k+1}, a_{i+k+1}, \ldots, a_n)$ whenever n-tuple $(a_1, a_2, \ldots, a_n)$ satisfies $a_i = a_{i+1} = \cdots = a_{i+k}, 1 \leq i,\ i + k \leq n$.

Let G be an arbitrary connected DNG. It is easy to check that the partition of the vertex set of G into non-empty subsets $U_1, U_2, \ldots, U_h, V_1, V_2, \ldots, V_h$ determines a divisor H of G. The $2h \times 2h$ adjacency matrix A_H has the following form:

$$A_H = \left[\begin{array}{c|c} \mathbf{0} & \begin{matrix} n_1 & n_2 & \ldots & n_{h-1} & n_h \\ n_1 & n_2 & \ldots & n_{h-1} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ n_1 & 0 & \ldots & 0 & 0 \end{matrix} \\ \hline \begin{matrix} m_1 & m_2 & \ldots & m_{h-1} & m_h \\ m_1 & m_2 & \ldots & m_{h-1} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ m_1 & 0 & \ldots & 0 & 0 \end{matrix} & \mathbf{0} \end{array}\right]$$

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Theorem 4.1. *Let λ be a nonzero eigenvalue of the connected DNG G, and let H be the divisor of G. Then λ is an eigenvalue of H.*

Proof. There are exactly $2h$ eigenvalues of G that belong to its divisor as well; the remaining are non-main and correspond to the sets of duplicate vertices. Therefore, each of them is equal to zero. ☐

The following corollary is an immediate consequence of the previous theorem.

Corollary 4.2. *Let G be an arbitrary DNG, H its divisor and let $k \in \mathbb{R}$, $k > 0$. Then*

(i) $\lambda_2(G) \le k$ *if and only if* $\lambda_2(H) \le k$.
(ii) $P_G(k) \le 0$ *(resp.* $P_G(k) > 0$*) if and only if* $P_H(k) \le 0$ *(resp.* $P_H(k) > 0$*).*

This corollary enables us to consider the spectrum of the divisor H of graph G instead of the spectrum of G itself. It is easy to see that if $P_H(\sqrt{2}) > 0$ holds, then the second largest eigenvalue of H is greater than $\sqrt{2}$, and thus the second largest eigenvalue of G is greater than $\sqrt{2}$, as well. Moreover, we get the following lemma.

Lemma 4.3. *Let $G = DNG(m_1, \dots, m_i, \dots, m_h; n_1, \dots, n_j, \dots, n_h)$ be a DNG, H be its divisor and $k \in \mathbb{R}$, $k > 0$. Then*

(i) If $\lambda_2(DNG(m_1, \dots, m_{i-1}, 1, m_{i+1}, \dots, m_h; n_1, \dots, n_h)) < k$, *and* $P_H(k) < 0$ *for every* $m_i \in \mathbb{N}$ *then*

$$\lambda_2(DNG(m_1, \dots, m_{i-1}, m_i, m_{i+1}, \dots, m_h; n_1, \dots, n_h)) < k$$

for every $m_i \in \mathbb{N}$.

(ii) If

$$\lambda_2(DNG(m_1, \dots, m_{i-1}, 1, m_{i+1}, \dots, m_h, n_1, \dots, n_{j-1}, 1, n_{j+1}, \dots, n_h)) < k,$$

and $P_H(k) < 0$ *for every* $m_i, n_j \in \mathbb{N}$ *then*

$$\lambda_2(DNG(m_1, \dots, m_i, m_{i+1}, \dots, m_h, n_1, \dots, n_{j-1}, n_j, n_{j+1}, \dots, n_h)) < k$$

for every $m_i, n_j \in \mathbb{N}$.

Proof. (i) By Corollary 4.2, $P_H(k) < 0$ implies $P_G(k) < 0$. Assume to the contrary, i.e., $\lambda_2(\mathrm{DNG}(m_1, \dots, m_{i-1}, m_i - 1, m_{i+1}, \dots, m_h; n_1, \dots, n_h)) < k$, but $\lambda_2(\mathrm{DNG}(m_1, \dots, m_{i-1}, m_i, m_{i+1}, \dots, m_h; n_1, \dots, n_h)) \ge k$. First, if

$$\lambda_2(\mathrm{DNG}(m_1, \dots, m_{i-1}, m_i, m_{i+1}, \dots, m_h; n_1, \dots, n_h)) = k,$$

then we get $P_H(k) = P_G(k) = 0$ contrary to assumption. Furthermore,

$$\lambda_2(\mathrm{DNG}(m_1, \dots, m_{i-1}, m_i, m_{i+1}, \dots, m_h; n_1, \dots, n_h)) > k$$

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and $P_G(k) < k$ imply that there are at least 3 eigenvalues of

$$\mathrm{DNG}(m_1, \ldots, m_{i-1}, m_i, m_{i+1}, \ldots, m_h; n_1, \ldots, n_h)$$

greater than k. Also, at most one eigenvalue of its vertex-deleted subgraph

$$\mathrm{DNG}(m_1, \ldots, m_{i-1}, m_i - 1, m_{i+1}, \ldots, m_h; n_1, \ldots, n_h)$$

satisfies this condition which is impossible by the Interlacing Theorem (see [4, Theorem 0.10]). This is a contradiction.

Applying the similar reasoning but on two parameters m_i and n_j we easily get the statement (ii). ☐

The next lemma gives an upper bound on the parameter h in a connected DNG satisfying $\lambda_2 \leq \sqrt{2}$.

LEMMA 4.4. *Let $G = DNG(m_1, \ldots, m_h; n_1, \ldots, n_h)$ be a connected DNG with $\lambda_2 \leq \sqrt{2}$. Then $h \leq 6$.*

Proof. Suppose $h > 6$. Consider an induced subgraph G' of G, obtained by deleting the cells $U_2, \ldots, U_{h-3}$, $V_3, \ldots, V_{h-4}$ and all but one vertex in each one of the remaining ten cells. Direct computation shows that the second largest eigenvalue of the graph G' is greater than $\sqrt{2}$ and thus (by the Interlacing Theorem) the second largest eigenvalue of the graph G is greater than $\sqrt{2}$, as well. ☐

Now we determine all connected DNGs with $\lambda_2 \leq \sqrt{2}$. Due to Lemma 4.4, $\lambda_2(G) \leq \sqrt{2}$ implies $h \leq 6$. So, naturally, we consider all possible values of h. The property $\lambda_2(G) \leq \sqrt{2}$ is a hereditary one (meaning that if a graph G has that property, that is also a property of each induced subgraph of G). If it occurs that a graph G has a given hereditary property, but at the same time no supergraph of G possesses it, then G is called a *maximal graph* for the observed property.

Clearly, we have $\lambda_2 \leq \sqrt{2}$ for any DNG satisfying $h = 1$ (the second largest eigenvalue of a complete bipartite graph is equal to zero). Now we consider the next case.

THEOREM 4.5. *Let $G = DNG(m_1, m_2; n_1, n_2)$ be a connected DNG satisfying $\lambda_2(G) \leq \sqrt{2}$. Then G is an induced subgraph of one of the graphs* 1-26 *given in Table* 1.

Proof. We compute that

$$P_H(\sqrt{2}) = 4 - 2(n_1 m_2 + m_1 n_1 + m_1 n_2) + m_1 n_1 m_2 n_2. \tag{4.1}$$

Putting $m_2 = 2$, $n_2 = 1$ in this expression we get $P_H(\sqrt{2}) = 4 - 4n_1 - 2m_1$, so in this case $P_H(\sqrt{2}) < 0$ for every $m_1, n_1 \in \mathbb{N}$. We can also check that $\lambda_2(\mathrm{DNG}(1, 2; 1, 1)) \leq$

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$\sqrt{2}$ holds. So, according to Lemma 4.3, we have $\lambda_2(\mathrm{DNG}(m_1, 2; n_1, 1)) \leq \sqrt{2}$ for every $m_1, n_1 \in \mathbb{N}$. The corresponding family of graphs is represented as family G_1 of Table 1. Furthermore, the parameters m_2 and n_2 cannot be increased if $m_1 \geq 15$ and $n_1 \geq 3$. That fact is easily checked by direct calculation of the spectra of $\mathrm{DNG}(15, 3; 3, 1)$ and $\mathrm{DNG}(15, 2; 3, 2)$ – for these two graphs $\lambda_2 > \sqrt{2}$. The similar application of Lemma 4.3 leads us to the conclusion that families of graphs G_2 and G_3 of Table 1 also satisfy the condition $\lambda_2 < \sqrt{2}$, for every $m_1, n_2 \in \mathbb{N}$; G_2 having $P_H(\sqrt{2}) = -2m_1$, and G_3 having $P_H(\sqrt{2}) = -4m_1$. Again, it is forbidden (for sufficiently large values of m_1 and n_2) to increase the values of the parameters m_2 and n_1 if we want to keep the property $\lambda_2 \leq \sqrt{2}$ in families G_2 and G_3. Graph G_4 of Table 1 gives us the upper bound on parameters m_2 and n_2, that is, G_4 has $\lambda_2 = \sqrt{2}$, and it is maximal (checked by direct computation).

Three families of graphs, and one finite maximal graph described above determine the boundaries within which we are going to find the rest of the maximal DNGs (or families of DNGs) that satisfy the condition $\lambda_2 \leq \sqrt{2}$.

We start from the family G_1, by increasing the parameter m_2, and letting $n_2 = 1$, but if $m_1 \leq 2$ then for $n_1 \in \mathbb{N}$, and $m_2 \in \mathbb{N}$ we get graphs of family G_3 (if $m_1 = 2$) or subgraphs of graphs belonging to family G_2 (if $m_1 = 1$). So, we have $m_1 \geq 3$, $n_2 = 1$, and $m_2 \geq 3$. By putting $m_1 = 3, n_1 = n_2 = 1$ into (4.1) we get $P_H(\sqrt{2}) = m_1 - 8$, and thus we also have $m_1 \leq 8$. Finally, cases to be considered arise: $(m_2, n_2) \in \{(i, 1),\ i = 3, \ldots, 8\}$:

(i) $m_2 = 3, n_2 = 1$, $P_H(\sqrt{2}) = (m_1 - 6)(n_1 - 2) - 8$. If $3 \leq m_1 \leq 6$, $P_H(\sqrt{2}) < 0$ holds for every $n_1 \in \mathbb{N}$, and $\lambda_2(\mathrm{DNG}(6, 3; 1, 1)) < \sqrt{2}$, so we have family $G_5 = \mathrm{DNG}(6, 3; n_1, 1)$ of Table 1. If $n_1 \leq 2$, $P_H(\sqrt{2}) < 0$ holds for every $m_1 \in \mathbb{N}$, and $\lambda_2(\mathrm{DNG}(1, 3; 2, 1)) < \sqrt{2}$, so we have family $G_6 = \mathrm{DNG}(m_1, 3; 2, 1)$ of Table 1. If $m_1 \geq 15$ and $n_1 \geq 3$ we get $P_H(\sqrt{2}) \geq 1$, so the (finite) maximal graphs are within those boundaries. We have:

- (i.1) $m_1 = 7$, and then $P_H(\sqrt{2}) = n_1 - 10$, so we get the graph $G_7 = \mathrm{DNG}(7, 3; 10, 1)$;
- (i.2) $m_1 = 8$, and then $P_H(\sqrt{2}) = 2n_1 - 12$, so we get the graph $G_8 = \mathrm{DNG}(8, 3; 6, 1)$;
- (i.3) $n_1 = 4$, and then $P_H(\sqrt{2}) = 2m_1 - 20$, so we get the graph $G_9 = \mathrm{DNG}(10, 3; 4, 1)$;
- (i.4) $n_1 = 3$, and then $P_H(\sqrt{2}) = m_1 - 14$, so we get the graph $G_{10} = \mathrm{DNG}(14, 3; 3, 1)$.

Graphs G_7–G_{10} of Table 1 all have $\lambda_2 = \sqrt{2}$, a fact easily checked by direct computation.

(ii) $m_2 = 4, n_2 = 1$, $P_H(\sqrt{2}) = 2((m_1 - 4)(n_1 - 1) - 2)$. If $3 \leq m_1 \leq 4$, $P_H(\sqrt{2}) < 0$ holds for every $n_1 \in \mathbb{N}$, and $\lambda_2(\mathrm{DNG}(4, 4; 1, 1)) < \sqrt{2}$, so we

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have family $G_{11} = \mathrm{DNG}(4,4;n_1,1)$ of Table 1. If $n_1 = 1$ $P_H(\sqrt{2}) < 0$ holds for every $m_1 \in \mathbb{N}$, and $\lambda_2(\mathrm{DNG}(1,4;1,1)) < \sqrt{2}$, so we have family $G_{12} = \mathrm{DNG}(m_1,4;1,1)$ of Table 1. If $m_1 \geq 7$ and $n_1 \geq 2$ we get $P_H(\sqrt{2}) \geq 1$, so the (finite) maximal graphs are within those boundaries. The remaining cases are $m_1 = 5$ and $m_1 = 6$:

(ii.1) $m_1 = 5$, and then $P_H(\sqrt{2}) = 2(n_1 - 3)$, so we get the graph $G_{13} = \mathrm{DNG}(5,4;3,1)$;

(ii.2) $m_1 = 6$, and then $P_H(\sqrt{2}) = 2(2n_1 - 4)$, so we get the graph $G_{14} = \mathrm{DNG}(6,4;2,1)$.

Again, graphs G_{13} and G_{14} have $\lambda_2 = \sqrt{2}$.

(iii) $m_2 = 5, n_2 = 1$, in this case, we can see that graph $\mathrm{DNG}(7,5;1,1)$ is forbidden for the property $\lambda_2 \leq \sqrt{2}$, so we have $3 \leq m_1 \leq 6$, and the possible cases are:

(iii.1) $m_1 = 3$, and then $P_H(\sqrt{2}) = -2 - n_1$, and $\lambda_2(\mathrm{DNG}(3,5;1,1)) < \sqrt{2}$ holds, so we get the family $\mathrm{DNG}(3,5;n_1,1)$, but each graph of that family is an induced subgraph of the corresponding graph of the family $G_{17} = \mathrm{DNG}(3,6;n_1,1)$;

(iii.2) $m_1 = 4$, and then $P_H(\sqrt{2}) = 2n_1 - 4$, so we get the graph $G_{15} = \mathrm{DNG}(4,5;2,1)$;

(iii.3) $m_1 = 5$ and $m_1 = 6$, but since $n_1 = 1$, in both cases we get $P_H(\sqrt{2}) = m_1 - 6$, and the resulting maximal graph is $G_{16} = \mathrm{DNG}(6,5;1,1)$.

For graphs G_{15} and G_{16}, $\lambda_2 = \sqrt{2}$ holds.

(iv) $m_2 = 6, n_2 = 1$, now $\mathrm{DNG}(5,6;1,1)$ is forbidden, and so $3 \leq m_1 \leq 4$. For $m_1 = 3$ we have $P_H(\sqrt{2}) = -2$, and $\lambda_2(\mathrm{DNG}(3,6;1,1)) < \sqrt{2}$, so we have family $G_{17} = \mathrm{DNG}(3,6;n_1,1)$ of Table 1, and for $m_1 = 4$, $P_H(\sqrt{2}) = 4n_1 - 4$ giving the graph $G_{18} = \mathrm{DNG}(4,6;1,1)$.

(v) $m_2 = 7, n_2 = 1$, $\mathrm{DNG}(4,7;1,1)$ is forbidden, and the only possibility is $m_1 = 3$. If so, $P_H(\sqrt{2}) = n_1 - 2$ giving the graph $G_{19} = \mathrm{DNG}(3,7;2,1)$.

(vi) $m_2 = 8, n_2 = 1$, again $m_1 \leq 3$ holds, but now $\mathrm{DNG}(3,8;2,1)$ is forbidden, so the only possibility is the graph $G_{20} = \mathrm{DNG}(3,8;1,1)$.

For graphs $G_{18} - G_{20}$ $\lambda_2 = \sqrt{2}$ holds. Let now $n_2 = 2$. Then $m_1 \geq 2$ holds (otherwise, we get the family G_2). Also, if $n_2 = 2$ and $m_1 \geq 2$ holds, then $m_2 \leq 4$, because $\mathrm{DNG}(2,5;1,2)$ is forbidden for the property $\lambda_2 \leq \sqrt{2}$. Three cases arise:

(i) $m_2 = 2, n_2 = 2$, $P_H(\sqrt{2}) = 2((m_1 - 2)(n_1 - 2) - 2)$, so if $m_1 \geq 5$ and $n_1 \geq 3$, $P_H(\sqrt{2}) \geq 1$. By inspecting all the possibilities, within the given range, we obtain one family $G_{21} = \mathrm{DNG}(m_1,2;2,2)$ and one maximal graph $G_{22} = \mathrm{DNG}(3,2;4,2)$ ($\lambda_2(G_{22}) = \sqrt{2}$).

(ii) $m_2 = 3, n_2 = 2$, $P_H(\sqrt{2}) = 2((2m_1 - 3)(n_1 - 1) - 1)$, so if $m_1 \geq 3$ and $n_1 \geq 2$, $P_H(\sqrt{2}) \geq 4$. By inspecting all the possibilities within the given frame we obtain one family $G_{23} = \mathrm{DNG}(m_1,3;1,2)$ and one maximal graph $G_{24} = \mathrm{DNG}(2,3;2,2)$ ($\lambda_2(G_{24}) = \sqrt{2}$).

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(iii) $m_2 = 4, n_2 = 2$, and graph $\mathrm{DNG}(3,4;1,2)$ is forbidden, so $m_1 = 2$ must hold. Then we have $P_H(\sqrt{2}) = 4n_1 - 4$, and this gives us one maximal graph $G_{25} = \mathrm{DNG}(2,4;1,2)$ with $\lambda_2 = \sqrt{2}$.

Let now $n_2 = 3$. The graph G_4 is maximal, so either $m_2 = 3$, or $m_2 = 4$. The case $m_2 = 4$ is giving us exactly the graph G_4. If we put $n_2 = m_2 = 3$ we get $m_1 \leq 2$ ($n_1 \leq 2$), and the graph $\mathrm{DNG}(2,3;2,3)$ is forbidden, so we have one maximal graph in this case: $G_{26} = \mathrm{DNG}(2,3;1,3)$.

We have exhausted all the possibilities. □

Note that each DNG with $h = 1$ is an induced subgraph of G_1 of Table 1 (i.e., it is not maximal for $\lambda_2 \leq \sqrt{2}$). Now we proceed with $h = 3$.

THEOREM 4.6. *Let $G = DNG(m_1, m_2, m_3; n_1, n_2, n_3)$ be a connected DNG satisfying $\lambda_2(G) \leq \sqrt{2}$. Then G is an induced subgraph of one of the graphs* 1-69 *given in Table* 2.

Proof. Again, we compute the value of the characteristic polynomial of the divisor H of G at $\lambda = \sqrt{2}$:

$$\begin{aligned} P_H(\sqrt{2}) = 8 &- 4n_1m_1 - 4n_1m_2 - 4n_1m_3 - 4n_3m_1 - 4n_2m_1 - 4n_2m_2 \\ &+2n_3m_1n_1m_2 + 2n_3m_1n_1m_3 + 2n_2m_1n_1m_3 + 2n_1m_2n_2m_3 \\ &+2n_3m_2n_2m_1 - n_3m_2n_2m_3m_1n_1. \end{aligned}$$

By analyzing this polynomial we determine infinite families of DNGs satisfying $\lambda_2(G) \leq \sqrt{2}$. Using them we set the boundaries for further investigation. These families are:

(i) $G_1 = \mathrm{DNG}(m_1,1,2;1,n_2,2)$, $m_1, n_2 \in \mathbb{N}$, $P_H(\sqrt{2}) = -4$
(ii) $G_2 = \mathrm{DNG}(m_1,2,1;2,n_2,1)$, $m_1, n_2 \in \mathbb{N}$, $P_H(\sqrt{2}) = -16$
(iii) $G_3 = \mathrm{DNG}(m_1,1,1;2,n_2,2)$, $m_1, n_2 \in \mathbb{N}$, $P_H(\sqrt{2}) = -8$
(iv) $G_4 = \mathrm{DNG}(m_1,2,2;1,n_2,1)$, $m_1, n_2 \in \mathbb{N}$, $P_H(\sqrt{2}) = -8$
(v) $G_5 = \mathrm{DNG}(2,m_2,1;2,n_2,1)$, $m_2, n_2 \in \mathbb{N}$, $P_H(\sqrt{2}) = -16$
(vi) $G_6 = \mathrm{DNG}(1,m_2,1;1,n_2,2)$, $m_2, n_2 \in \mathbb{N}$, $P_H(\sqrt{2}) = -4$
(vii) $G_7 = \mathrm{DNG}(2,m_2,2;1,n_2,1)$, $m_2, n_2 \in \mathbb{N}$, $P_H(\sqrt{2}) = -8$

We can also set the boundaries on m_3 and n_3 by putting $m_1 = m_2 = n_1 = n_2 = 1$: $P_H(\sqrt{2}) = m_3n_3 - 8$. So, there are exactly eleven cases: $n_3 = 1$, $m_3 = 1, \ldots, 8$ and $n_3 = 2$, $m_3 = 2,3,4$. Examination (described in the proof of Theorem 4.5) of all possibilities in each one of these cases within the settled boundaries leads to the maximal finite graphs (or infinite families of graphs) given in Table 2. □

The next two theorems are proven in the similar way.

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THEOREM 4.7. *Let $G = DNG(m_1, m_2, m_3, m_4; n_1, n_2, n_3, n_4)$ be a connected DNG satisfying $\lambda_2(G) \leq \sqrt{2}$. Then G is an induced subgraph of one of the graphs* 1-77 *given in Table* 3.

THEOREM 4.8. *Let $G = DNG(m_1, \ldots, m_5; n_1, \ldots, n_5)$ be a connected DNG satisfying $\lambda_2(G) \leq \sqrt{2}$. Then G is an induced subgraph of one of the graphs $1 - 27$ given in Table* 4.

Finally, we prove the next result.

THEOREM 4.9. *The graph $G = DNG(1^6; 1^6)$ is a unique DNG satisfying $h = 6$ and $\lambda_2(G) \leq \sqrt{2}$.*

Proof. By direct computation, we get $\lambda_2(G) < \sqrt{2}$. Also we get that the increasing of any of the parameters which describe G implies $\lambda_2(G) > \sqrt{2}$. □

Collecting the results above we arrive at the following theorem.

THEOREM 4.10. *Let G be an arbitrary DNG satisfying $\lambda_2(G) \leq \sqrt{2}$. Then G is an induced subgraph of one of the graphs given in Tables* 1-4 *or $G = DNG(1^6; 1^6)$.*

We conclude the section by an example of the previous technique applied in a simple case. There are some ways to determine all connected DNGs with $\lambda_2 \leq 1$. Namely, after determining all DNGs with the property $\lambda_2 \leq \sqrt{2}$, one could proceed and among them (and their subgraphs) find the ones that satisfy $\lambda_2 \leq 1$, but this includes searching a large number of graphs and some infinite families, as well. On the other hand, DNGs are bipartite, and the bipartite graphs satisfying $\lambda_2 \leq 1$ were characterized in 1991 by M. Petrović using the method of minimal forbidden subgraphs (see, for example [9, p. 53-57]). But again, the determination of DNGs with $\lambda_2 \leq 1$ using this result includes the comparison of every obtained forbidden subgraph to the graphs having double nested structure. Here we use the same method used for $\lambda_2 \leq \sqrt{2}$. It turns out that the whole procedure and the final result are much simpler.

EXAMPLE 4.11. *A connected DNG satisfying $\lambda_2(G) \leq 1$ is an induced subgraph of one of DNGs whose parameters are:*

- $(*, 1; 1, *), (*, 1; *, 1), (1, 2; 1, 2), (2^2; *, 1), (*, 2; 1^2), (3, 2; 2, 1), (2, 3; 1^2),$
- $(*, 1^2; 1, *, 1), (1, *, 1; 1, *, 1), (2^2, 1; 1, *, 1), (2, 1^2; 2, 1^2), (1, *, 2; 1^3),$
- $(1, *, 1^2; 1, *, 1^2)$

We get $h \leq 4$ (compare Lemma 4.4*).*

If $h = 2$ we have $P_H(1) = 1 - m_2n_1 - m_1n_1 - m_1n_2 + m_1n_1m_2n_2$. Putting $m_1 = n_2 = 1$ we get $P_H(1) = -m_1 < 0$ and applying Lemma 4.3 *gives the first solution. The second solution we get by putting $m_2 = n_2 = 1$ ($P_H(1) = 1 - m_1 - n_1$)*

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and applying Lemma 4.3*. The third solution we get by direct computation. Now let* $n_2 = 1$ *and* $m_2 \geq 2$*. If so,* $m_1 \geq 2$ *must hold (otherwise, we have our first solution for any choice of (positive) integers* n_1, m_2*). Also, if* $m_1 \geq 2$ *holds then* $m_2 \leq 3$ *must hold (we get that condition by direct computation). Therefore, we have the following cases depending on* m_2 *and* n_2*:*

(i) $m_2 = 2$, $n_2 = 1$, $P_H(1) = (m_1 - 2)(n_1 - 1) - 1$ *If* $m_1 = 2$ *and* $n_1 \in \mathbb{N}$ *the application of Lemma 4.3 gives the fourth solution. If* $n_1 = 1$ *and* $m_1 \in \mathbb{N}$ *the application of Lemma 4.3 gives the fifth solution. The sixth solution we get by direct checking that* $\lambda_2(G) \leq 1$ *holds for the graph* $G = DNG(3, 2; 2, 1)$*. If* $m_1 > 3$ *and* $n_1 \geq 2$ *or* $m_1 \geq 3$ *and* $n_1 > 2$*, then* $P_H(1) > 0$ *holds.*

(ii) $m_2 = 3$, $n_2 = 1$, *and now graphs* $DNG(3, 3; 1^2)$ *and* $DNG(2, 3; 2, 1)$ *are forbidden, and for the graph* $G = DNG(2, 3; 1^2)$, $\lambda_2(G) \leq 1$ *holds, so we have obtained the seventh solution.*

Proceeding in a similar way we consider the case $h = 3$ *and get the five listed solutions.*

Finally, by direct computation we get that the graphs $DNG(1^3, 2; 1^4)$, $DNG(1^2, 2, 1; 1^4)$ *and* $DNG(2, 1^3; 1^4)$ *are forbidden for the property* $\lambda_2(G) \leq 1$*. Also, we get* $P_H(1) = -m_2n_2 - 4(m_2 + n_2 + 2)$*, meaning that* $P_H(1) < 0$ *for any choice of positive integers* m_2 *and* n_2*. Application of Lemma* 4.3 *gives the last solution and concludes our consideration.*

5. Appendix. The following tables contain the representations of all maximal connected DNGs with $\lambda_2 \leq \sqrt{2}$ (obtained in Theorems 4.5-4.8).

G	m_1	m_2	n_1	n_2	G	m_1	m_2	n_1	n_2	G	m_1	m_2	n_1	n_2
1.	$*$	2	$*$	1	10.	14	3	3	1	19.	3	7	2	1
2.	$*$	2	1	$*$	11.	$*$	4	1	1	20.	3	8	1	1
3.	$*$	1	2	$*$	12.	4	4	$*$	1	21.	$*$	2	2	2
4.	1	4	1	3	13.	5	4	3	1	22.	3	2	4	2
5.	6	3	$*$	1	14.	6	4	2	1	23.	$*$	3	1	2
6.	$*$	3	2	1	15.	4	5	2	1	24.	2	3	2	2
7.	7	3	10	1	16.	6	5	1	1	25.	2	4	1	2
8.	8	3	6	1	17.	3	6	$*$	1	26.	2	3	1	3
9.	10	3	4	1	18.	4	6	1	1					

Table 1: Maximal double nested graphs with $h = 2$ satisfying $\lambda_2 \leq \sqrt{2}$.

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G	m_1	m_2	m_3	n_1	n_2	n_3	G	m_1	m_2	m_3	n_1	n_2	n_3
1.	$*$	1	2	1	$*$	2	36.	4	4	1	2	$*$	1
2.	$*$	2	1	2	$*$	1	37.	3	6	1	2	$*$	1
3.	$*$	1	1	2	$*$	2	38.	6	3	2	1	$*$	1
4.	$*$	2	2	1	$*$	1	39.	4	4	2	1	$*$	1
5.	2	$*$	1	2	$*$	1	40.	3	6	2	1	$*$	1
6.	1	$*$	2	1	$*$	2	41.	1	2	2	3	2	1
7.	2	$*$	2	1	$*$	1	42.	1	1	2	4	2	1
8.	6	1	1	$*$	1	1	43.	6	1	2	2	1	1
9.	7	1	1	38	1	1	44.	4	2	2	2	1	1
10.	8	1	1	22	1	1	45.	3	2	2	3	1	1
11.	9	1	1	16	1	1	46.	3	4	2	2	1	1
12.	10	1	1	14	1	1	47.	1	4	3	$*$	1	1
13.	11	1	1	12	1	1	48.	1	12	3	3	1	1
14.	4	2	1	$*$	1	1	49.	1	8	3	4	1	1
15.	5	2	1	10	1	1	50.	1	6	3	7	1	1
16.	6	2	1	6	1	1	51.	1	5	3	10	1	1
17.	8	2	1	4	1	1	52.	2	$*$	3	2	1	1
18.	12	2	1	3	1	1	53.	3	4	3	1	1	1
19.	4	3	1	6	1	1	54.	4	2	3	1	1	1
20.	3	4	1	$*$	1	1	55.	6	1	3	1	1	1
21.	3	5	1	6	1	1	56.	2	$*$	3	1	2	1
22.	3	2	1	10	2	1	57.	1	4	4	2	1	1
23.	4	2	1	6	2	1	58.	1	3	4	3	1	1
24.	3	3	1	8	2	1	59.	1	2	4	$*$	1	1
25.	4	3	1	4	2	1	60.	2	$*$	4	1	1	1
26.	3	4	1	6	2	1	61.	1	4	5	1	1	1
27.	3	5	1	4	2	1	62.	1	1	6	$*$	1	1
28.	3	4	1	4	3	1	63.	1	2	6	2	1	1
29.	3	5	1	3	4	1	64.	1	1	7	2	1	1
30.	7	3	1	1	8	1	65.	1	1	8	1	1	1
31.	8	3	1	1	4	1	66.	2	2	2	1	$*$	2
32.	10	3	1	1	2	1	67.	1	$*$	3	1	2	2
33.	14	3	1	1	1	1	68.	1	$*$	3	2	1	2
34.	5	4	1	1	1	1	69.	1	$*$	4	1	1	2
35.	6	3	1	2	$*$	1							

Table 2: Maximal double nested graphs with $h = 3$ satisfying $\lambda_2 \leq \sqrt{2}$.

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G	m_1	m_2	m_3	m_4	n_1	n_2	n_3	n_4
1.	2	$*$	2	1	2	$*$	1	1
2.	2	$*$	2	2	1	$*$	1	1
3.	2	$*$	1	2	1	$*$	2	1
4.	1	$*$	2	2	1	$*$	1	2
5.	$*$	1	1	1	1	4	1	1
6.	$*$	1	1	1	1	1	4	1
7.	$*$	1	1	1	1	2	2	1
8.	4	2	1	1	1	$*$	1	1
9.	5	2	1	1	1	8	1	1
10.	6	2	1	1	1	4	1	1
11.	8	2	1	1	1	2	1	1
12.	12	2	1	1	1	1	1	1
13.	4	3	1	1	1	4	1	1
14.	3	4	1	1	1	$*$	1	1
15.	3	5	1	1	1	4	1	1
16.	7	1	1	1	1	36	1	1
17.	8	1	1	1	1	20	1	1
18.	9	1	1	1	1	13	1	1
19.	10	1	1	1	1	12	1	1
20.	11	1	1	1	1	10	1	1
21.	12	1	1	1	1	9	1	1
22.	14	1	1	1	1	8	1	1
23.	16	1	1	1	1	7	1	1
24.	22	1	1	1	1	6	1	1
25.	38	1	1	1	1	5	1	1
26.	3	4	1	1	2	$*$	1	1
27.	4	2	1	1	2	$*$	1	1
28.	6	1	1	1	2	$*$	1	1
29.	3	1	1	1	3	1	1	1
30.	3	2	2	1	1	1	1	1
31.	1	2	2	1	$*$	1	1	1
32.	1	3	2	1	10	1	1	1
33.	1	4	2	1	6	1	1	1
34.	1	6	2	1	4	1	1	1
35.	1	10	2	1	3	1	1	1
36.	1	1	2	1	10	2	1	1
37.	1	2	2	1	3	5	1	1
38.	1	4	2	1	3	4	1	1
39.	1	6	2	1	3	3	1	1

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40.	1	8	2	1	3	2	1	1
41.	1	4	2	1	4	2	1	1
42.	1	2	2	1	4	3	1	1
43.	1	2	2	1	6	2	1	1
44.	1	12	3	1	1	1	1	1
45.	1	4	3	1	2	1	1	1
46.	1	2	3	1	6	1	1	1
47.	1	1	3	1	8	2	1	1
48.	1	5	3	1	1	8	1	1
49.	1	6	3	1	1	4	1	1
50.	1	8	3	1	1	2	1	1
51.	1	4	3	1	2	$*$	1	1
52.	1	2	3	1	3	4	1	1
53.	1	2	3	1	4	2	1	1
54.	1	3	4	1	1	1	1	1
55.	1	2	4	1	2	1	1	1
56.	1	1	4	1	3	5	1	1
57.	1	1	4	1	4	3	1	1
58.	1	1	4	1	6	2	1	1
59.	1	2	4	1	2	$*$	1	1
60.	1	1	5	1	3	4	1	1
61.	1	1	5	1	4	2	1	1
62.	1	1	6	1	2	$*$	1	1
63.	1	2	2	1	1	2	2	1
64.	6	1	1	2	1	$*$	1	1
65.	4	2	1	2	1	$*$	1	1
66.	3	4	1	2	1	$*$	1	1
67.	1	4	1	2	2	1	1	1
68.	1	1	2	1	3	1	1	1
69.	1	2	2	2	2	1	1	1
70.	1	1	2	2	3	1	1	1
71.	1	4	3	2	1	$*$	1	1
72.	1	2	4	2	1	$*$	1	1
73.	1	1	4	2	2	1	1	1
74.	1	1	6	2	1	$*$	1	1
75.	1	1	4	3	1	1	1	1
76.	1	2	2	3	1	1	1	1
77.	1	4	1	3	1	1	1	1

Table 3: Maximal double nested graphs with $h = 4$ satisfying $\lambda_2 \leq \sqrt{2}$.

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G	m_1	m_2	m_3	m_4	m_5	n_1	n_2	n_3	n_4	n_5
1.	1	36	1	1	1	1	5	1	1	1
2.	1	20	1	1	1	1	6	1	1	1
3.	1	14	1	1	1	1	7	1	1	1
4.	1	12	1	1	1	1	8	1	1	1
5.	1	10	1	1	1	1	9	1	1	17
6.	2	$*$	1	1	1	1	4	1	1	1
7.	3	1	1	1	1	1	1	1	1	1
8.	1	2	2	1	1	2	$*$	1	1	1
9.	1	3	2	1	1	1	8	1	1	1
10.	1	4	2	1	1	1	4	1	1	1
11.	1	6	2	1	1	1	2	1	1	1
12.	1	8	2	1	1	1	1	1	1	1
13.	1	2	3	1	1	1	4	1	1	1
14.	1	1	4	1	1	2	$*$	1	1	1
15.	1	1	5	1	1	1	4	1	1	1
16.	1	8	2	1	1	1	1	2	1	1
17.	1	4	2	1	1	1	2	2	1	1
18.	1	2	3	1	1	1	2	2	1	1
19.	1	1	3	1	1	1	6	2	1	1
20.	1	1	4	1	1	1	4	2	1	1
21.	1	1	5	1	1	1	2	2	1	1
22.	1	2	4	1	1	1	1	3	1	1
23.	1	1	5	1	1	1	1	4	1	1
24.	1	1	2	2	1	1	1	1	1	1
25.	1	4	1	1	2	1	$*$	1	1	1
26.	1	2	2	1	2	1	$*$	1	1	1
27.	1	1	4	1	2	1	$*$	1	1	1

Table 4: Maximal double nested graphs with $h = 5$ satisfying $\lambda_2 \leq \sqrt{2}$.

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