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THE NESTED SPLIT GRAPHS WHOSE SECOND LARGEST EIGENVALUE IS EQUAL TO 1[1]

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Abstract. We determine all nested split graphs (NSG for short; i.e. graphs having no induced subgraphs equal to $2K_2$, P_4, or C_4) having the second largest eigenvalue equal to 1 and give some data regarding obtained results. The initial results in this research are given in the previous work of the second author, where all NSGs whose second largest eigenvalue is less than 1 are determined. It turns out that this case is well complicated with a number of solutions including some infinite families.

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1. Introduction

We consider only simple graphs, that is finite undirected graphs without loops or multiple edges. If G is such a graph with the vertex set $V_G = \{v_1, v_2, \dots, v_n\}$, the *adjacency matrix* of G is the $n \times n$ matrix $A_G = (a_{ij})$, where $a_{ij} = 1$ if there is an edge between the vertices i and j, and 0 otherwise. A *characteristic polynomial* of G is the characteristic polynomial of its adjacency matrix, so $P_G(\lambda) = \det(\lambda I - A_G)$, while the *eigenvalues* of G, denoted by

$$\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G),$$

are just the eigenvalues of A_G. In the sequel, we will usually suppress graph name from our notation.

Note, the eigenvalues of G are real and do not depend on vertex labelling. Additionally, for the connected graphs $\lambda_1 > \lambda_2$ holds. The eigenvalue λ_1 is known as a *graph index*. For more details on graph spectra see [4].

The problem of determining the graphs whose second largest eigenvalue does not exceed 1 was posed in [3]. Basic properties of these graphs are presented in the same paper. In the subsequent years, many results concerning this problem are obtained. These results will not be listed here, but one can consult some of papers [6–8, 10, 11], or their references.

The graphs having no induced subgraphs equal to $2K_2$, P_4, or C_4 are called (by P. Hansen) *nested split graphs*, or NSGs for short. These graphs play an important role in the investigations concerning the graphs with maximal index.

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Namely, it is known that the graph with maximal index and fixed size is an NSG (see, for example, [9, Theorem 2.2]). In [10], the NSGs whose second largest eigenvalue is less than 1 are determined. Here we complete this research by determining the graphs as in the title.

The paper is organized as follows. In Section 2 we recall some necessary results from [10], and we also mention some results from the literature in order to make the paper more self–contained. In Section 3 we determine all NSGs with $\lambda_2 = 1$. All such connected graphs are listed in a separate table.

2. Preliminaries

In the similar way as in [10], we list some notation and results in order to make the paper more self–contained.

Each P_4–free graph (i.e. *cograph*) can be represented by a *cotree*. This representation is explained in [2], while its modification is given in [1]. Here we present the main ideas from [1].

Let T_G be a cotree, representing a cograph G. In what follows $\oplus$ and $\otimes$ stand for the (disjoint) union and join of two graphs. The cotree T_G is a rooted tree in which any interior vertex w is either of $\oplus$–type (corresponds to union) or $\otimes$–type (corresponds to join). The terminal vertices (leaves) are typeless (each of them represents itself in G). Any interior vertex, say w, represents a subgraph of G induced by the terminal successors of w, and it is denoted by G_w. The direct successor (or a child) of any interior vertex w has a type which differs from the type of w (or it is typeless if being the terminal vertex). In addition, each non–terminal vertex has at least two children. Note also that, in this way, all internal vertices of any path from the root to any terminal vertex is $(\otimes, \oplus)$–alternating. It is worth mentioning that described representation is unique.

Apparently, each NSG is a cograph, and so we can use the same concept for its representation. It is proved in [10] that if T_G is a representation of an arbitrary NSG G, then each non–terminal vertex of T_G has at most one non–terminal direct successor. Due to this result it is sufficient to say if G is connected or not (note, G is connected if and only if the root of T_G is of $\otimes$–type) and to list the numbers of terminal successors of each non–terminal vertex of T_G (in natural order). Therefore, we use $C(a_1, a_2, \ldots, a_n)$ to denote an NSG such that the tree $T_{C(a_1,a_2,\ldots,a_n)}$ has exactly n non–terminal vertices, while its root is of $\otimes$–type and has exactly a_1 direct terminal successors; non–terminal successor of the root has exactly a_2 direct terminal successors, etc. A disconnected NSG is denoted by $D(a_1, a_2, \ldots, a_n)$.

Recall that each non–terminal vertex has at least two direct successors. Thus, we will assume that $a_1, a_2, \ldots, a_{n-1}$ are positive integers, while $a_n \geq 2$. (Note that $X(a_1, a_2, \ldots, a_{n-1}, 1)$ and $X(a_1, a_2, \ldots, a_{n-1} + 1)$ represent two isomorphic NSGs, where X stands for either C or D.) If for n–tuple $(a_1, a_2, \ldots, a_n)$ holds $a_i = a_{i+1} = \cdots = a_{i+k}, 1 \leq i,\ i + k \leq n$, we write $(a_1, a_2, \ldots, a_i^{k+1}, a_{i+k+1}, \ldots, a_n)$.

Since each NSG has at most one non–trivial component (called a dominate component, which is an NSG, as well) its second largest eigenvalue is equal to the second largest eigenvalue of a dominate component. Thus, it is sufficient to consider the connected NSGs since each disconnected NSG is obtained by adding the isolated vertices to connected one.

We now focus our attention to so–called *divisor concept.* Given an $s \times s$ matrix $D = (d_{ij})$, let the vertex set of a graph G be partitioned into non–empty subsets $V_1, V_2, \dots, V_s$ so that for any $i, j \in \{1, 2, \dots, s\}$ each vertex from V_i is adjacent to exactly d_{ij} vertices of V_j. The multidigraph H with the adjacency matrix D is called a *front divisor* of G, or briefly, a *divisor* of G, see [5, Definition 2.4.4].

Let $G = C(a_1, a_2, \dots, a_n)$ be an arbitrary connected NSG, and let V_i denote the set of vertices corresponding to a_i $(i = 1, 2, \dots, n)$. Hence, $|V_i| = a_i$ $(i = 1, 2, \dots, n)$. It is easy to check that the partition of vertex set of G into the non–empty subsets $V_1, V_2, \dots, V_n$ determines a divisor H of G. The $n \times n$ adjacency matrix D of H has the following form:

$$D = \begin{pmatrix} a_1 - 1 & a_2 & a_3 & a_4 & \dots & a_n \\ a_1 & 0 & 0 & 0 & \dots & 0 \\ a_1 & 0 & a_3 - 1 & a_4 & \dots & a_n \\ a_1 & 0 & a_3 & 0 & \dots & 0 \\ \vdots & & \ddots & & & \vdots \end{pmatrix}. \tag{2.1}$$

The following result can be proved in the same way as the corresponding result from [10] (see Theorem 1.1 and Corollary 1.1); the only difference makes '=' instead of '<' in its formulation.

Lemma 2.1. *Let G be an arbitrary NSG and let H be its divisor. Then, $\lambda_2(G) = 1$ if and only if $\lambda_2(H) = 1$.*

3. Main results

We proceed to determine all connected NSGs with $\lambda_2 = 1$. In further, let $G = C(a_1, a_2, \dots, a_n)$ be an arbitrary connected NSG, and let D denote the adjacency matrix of its divisor (having the form (2.1)).

First, we prove the following general result concerning NSGs with $\lambda_2 \le 1$.

Lemma 3.1. *Let $C(a_1, a_2, \dots, a_n)$ be a connected NSG with $\lambda_2 = 1$. Then $n \le 9$ holds.*

Proof. If $n > 10$ then an NSG $C(a_1, a_2, \dots, a_n)$ contains $C(1^{10}, 2)$ as an induced subgraph. Since the second largest eigenvalue of $C(1^{10}, 2)$ is greater than 1 (this can be computed directly), by the Interlacing Theorem (see, for example, [4, p. 19]) we get that the second largest eigenvalue of $C(a_1, \dots, a_n)$ is greater than 1. Furthermore, there are no solutions in the case $n = 10$, which can be easily checked by considering the NSG $C(1^9, 2)$ and its vertex–added supergraphs, and the proof is complete. ☐

Due to the previous lemma we have to consider all possible values of n $(n \le 9)$. Due to Lemma 2.1, if the equality $\det(I - D) = 0$ holds, then $\lambda_2(G) = 1$.

We prove a theorem.

Theorem 3.2. *Let* $G = C(a_1, a_2, \dots, a_n)$ *be a connected NSG satisfying* $\lambda_2(G) = 1$. *If* $n \leq 4$ *then* G *is some of the following graphs:* $C(a_1, a_2, a_3)$, *where* $(a_1, a_2, a_3) = (3, 1, 8)$, $(1, 2, 7)$, $(4, 1, 6)$, $(6, 1, 5)$, $(1, 3, 4)$, $(2^2, 4)$, $(1, 4, 3)$, *or* $(2, 3^2)$; $C(a_1, a_2, a_3, a_4)$, *where* $(a_1, a_2, a_3, a_4) = (14, 1, 3, 2)$, $(10, 1, 3, 3)$, $(8, 1, 3, 5)$, $(7, 1, 3, 9)$, $(5, 1, 4, 2)$, $(4, 2^3)$, $(3, 2^2, 3)$, $(1, 8, 1, 2)$, $(2, 7, 1, 2)$, $(1, 6, 1, 3)$, $(1, 5, 1, 5)$, $(2, 5, 1, 3)$, $(2, 4, 1, 5)$, $(3, 4, 1, 4)$, $(3, 3, 1, 13)$, $(4, 3, 1, 9)$, $(6, 3, 1, 7)$, *or* $(10, 3, 1, 6)$.

Proof. For $n = 1$ we get a complete graph, and the second largest eigenvalue of any complete graph is less than 1. Further, any NSG having the form $G = C(a_1, a_2)$, $(a_1 \geq 1, a_2 \geq 2)$ is a complete multipartite graph, and therefore it has exactly one positive eigenvalue (see [8], or [4, Theorem 2.4, pp. 73-74]). Consequently, $\lambda_2(G) < 1$ holds for every integers $a_1 \geq 1, a_2 \geq 2$.

Assume that $n = 3$. Let H be the divisor of G whose matrix has the form (2.1). We get $\det(I - D) = (2 - a_1a_2)(2 - a_3) - 2a_1$.

Now we have to check for which parameters a_i $(i = 1, \dots, 4)$ this determinant is equal to zero (since in these cases we have $\lambda_2(H) = 1$, and, by Lemma 2.1, $\lambda_2(G) = 1$). First, if $(a_1, a_2) = (1, 2)$ or $(a_1, a_2) = (2, 1)$, the determinant is negative for any choice of the third parameter. Similarly, if $a_3 = 2$ the determinant is negative for any choice of a_1 and a_2. Further for $a_3 = 8$ the determinant reduces to $2(a_1(3a_2 - 1) - 6)$, and it is positive for any choice of (a_1, a_2) satisfying $a_1 + a_2 \geq 4$ and $(a_1, a_2) \neq (3, 1)$. For $(a_1, a_2, a_3) = (3, 1, 8)$ the determinant is zero, so we have that solution.

The remaining cases are $a_3 = 3, 4, \dots, 7$. By computing the above determinant and inspecting whether it is equal to zero, we get the remaining solutions: $(a_1, a_2, a_3) = (1, 2, 7)$, $(4, 1, 6)$, $(6, 1, 5)$, $(1, 3, 4)$, $(4, 2, 2)$, $(1, 4, 3)$, and $(2, 3, 3)$.

Assume now that $n = 4$. Here we have

$$\det(I - D) = (a_1a_2a_3 - 2(a_1 + a_3))(a_4 + 1) - 2a_1a_2 + 4. \tag{3.1}$$

First, if $(a_1, a_2) = (1, 2)$ or $(a_1, a_2) = (2, 1)$, the determinant is negative for any choice of a_3 and a_4. Similarly, if $(a_2, a_3) = (1, 2)$ or $(a_2, a_3) = (2, 1)$ the determinant is negative for any choice of a_1 and a_4. By direct computation we get that the graph $C(2^2, 3, 2)$ has the second largest eigenvalue greater than 1. Hence, for $a_3 \leq 3$ and $a_1 + a_2 > 3$ and $a_2 \geq 2$ the determinant is positive for any choice of parameter a_4, and for $a_3 \leq 3$ and $a_1 + a_2 \leq 3$ the determinant is negative for any choice of the parameter a_4. The remaining cases are $a_3 \geq 3$ (when $a_2 = 1$), $a_3 = 2$, and $a_3 = 1$.

Case 1: $a_3 \geq 3$ and $a_2 = 1$. The inequality $a_3 \geq 7$ implies $a_1 + a_2 \leq 3$, and so we can conclude that in these cases the determinant is negative. In the case that $a_3 = 6$, or $a_3 = 5$ and $a_2 = 1$, we also conclude that the determinant is negative. We continue with $a_3 = 4$ and $a_2 = 1$. Here, by direct computation, we get that the second largest eigenvalue of the graph $C(6, 1, 4, 2)$ is greater than 1. Hence, we have $a_1 \leq 5$. If $a_1 = 5$ the solution is the graph $(a_1, a_2, a_3, a_4) = (5, 1, 4, 2)$, and for $a_1 = 1, 2, 3$, or 4 the determinant is negative for any choice of the remaining parameter.

Finally, if $a_3 = 3$ and $a_2 = 1$, similarly as above, we get that the second largest eigenvalue of the graph $C(15, 1, 3, 2)$ is greater than 1. Hence, we have $a_1 \le 14$. If $a_1 = 14$ the solution is $(a_1, a_2, a_3, a_4) = (14, 1, 3, 2)$. By inspecting other possibilities we get the following solutions: $(a_1, a_2, a_3, a_4) = (10, 1, 3, 3)$, $(8, 1, 3, 5)$, and $(7, 1, 3, 9)$.

Case 2: $a_3 = 2$. By direct computation, we get that the second largest eigenvalue of the graph $C(1, 4, 2, 2)$ is greater than 1. Hence, we get that $a_2 \le 3$. By putting $a_2 = 3, 2$, and 1 into (3.1) we get the solutions $(a_1, a_2, a_3, a_4) = (4, 2, 2, 2)$, and $(3, 2, 2, 3)$.

Case 3: $a_3 = 1$. By direct computation, we get that the second largest eigenvalue of the graph $C(1, 9, 1, 2)$ is greater than 1. Hence, $a_2 \le 8$, and for $a_2 = 8$ one solution is $(a_1, a_2, a_3, a_4) = (1, 8, 1, 2)$. By putting $a_2 = 7, 6, \ldots, 1$ into (3.1), we get the following solutions: $(a_1, a_2, a_3, a_4) = (2, 7, 1, 2), (1, 6, 1, 3)$, $(1, 5, 1, 5), (2, 5, 1, 3)$, $(2, 4, 1, 5), (3, 4, 1, 4)$, $(3, 3, 1, 13)$, $(4, 3, 1, 9)$, $(6, 3, 1, 7)$, and $(10, 3, 1, 6)$.

The proof is complete. □

The following lemma will be frequently used in the sequel.

Lemma 3.3. *Let $C(a_1, a_2, \ldots, a_n)$ be an NSG with $\lambda_2 = 1$. Then*

(i) if $n = 5$ we have $a_2 \le 4$, and $a_5 \le 8$;

(ii) if $n = 6$ we have $a_2 \le 3$, $a_4 \le 6$, and $a_5 \le 6$;

(iii) if $n = 7$ we have $a_1 \le 6$, $a_2 \le 2$, $a_4 \le 2$, $a_5 \le 6$, and $a_7 \le 3$;

(iv) if $n = 8$ we have $a_1 \le 3$, $a_2 \le 2$, $a_4 \le 2$, $a_5 \le 5$, $a_6 \le 5$, and $a_7 \le 3$;

(v) if $n = 9$ we have $a_1 = a_2 = a_4 = a_6 = a_7 = 1$, $a_3 \le 4$, $a_5 \le 4$, and $a_9 = 2$.

Proof. (i) By direct computation, we get that the graph $C(1, 5, 1^2, 2)$ has the second largest eigenvalue greater than 1. The same holds for the graph $C(1^4, 9)$. Hence, we have $a_2 \le 3$ and $a_5 \le 8$.

The remaining cases we prove in the similar way. □

We now prove a sequence of similarly formulated statements (Theorem 3.4 – Theorem 3.8).

Theorem 3.4. *If $C(a_1, a_2, \ldots, a_5)$ is a connected NSG satisfying $\lambda_2 = 1$ then we have $(a_1, a_2, a_3, a_4, a_5) = (1, 4, 1, k, 2)$, $(1, 3, 2, k, 2)$, $(2, 3, 1, k, 2)$, $(1, 2, k, 1, 4)$, $(1, 2, k, 2, 3)$, $(2^3, k, 2)$, $(1^4, 8)$, $(1^2, 2, 1, 6)$, $(1^2, 4, 1, 5)$, $(2, 1, k, 1, 4)$, $(2, 1, k, 2, 3)$, $(6, 1^3, 3)$, $(4, 1, 2, 1, 3)$, $(3, 1, 4, 1, 3)$, $(3, 1, 6, k, 2)$, $(4, 1, 4, k, 2)$, or $(6, 1, 3, k, 2)$, where $k \ge 1$.*

Proof. Due to Lemma 3.3 (i) we have that $a_2 \le 4$ and $a_5 \le 8$. If $a_2 = 4$, we get $a_1 = a_4 = 1$ and $a_5 = 2$. By putting the fixed values a_1, a_2, a_4 and a_5 into the determinant of (2.1), we get that the determinant is equal to zero for any choice of parameter a_3. Hence, the solution is $(a_1, a_2, a_3, a_4, a_5) = (1, 4, 1, k, 2)$, where $k \geq 1$.

We now distinguish three more cases depending on a_2.

Case 1: $a_2 = 3$. By direct computation, we get that the second largest eigenvalue of the graph $C(3, 3, 1, 1, 2)$ is greater than 1. Hence, $a_1 \leq 2$. We also get that the second largest eigenvalue of the graph $C(1, 3, 3, 1, 2)$, or $C(1, 3, 1, 1, 3)$ is greater than 1. Hence, $a_3 \leq 2$, and $a_5 = 2$. For $a_3 = 2$, we get $a_1 = 1$. By putting $a_1 = 1, a_2 = 3, a_3 = a_5 = 2$ into (2.1), we get that the determinant is zero for any choice of parameter a_4. Hence, the solution is $(a_1, a_2, a_3, a_4, a_5) = (1, 3, 2, k, 2)$, where $k \geq 1$.

Assume now that $a_3 = 1$. We get that $a_1 \leq 2$, and for $a_1 = 2$ we have the solution $(a_1, a_2, a_3, a_4, a_5) = (2, 3, 1, k, 2)$. If $a_1 = 1$ the corresponding determinant is negative for any choice of the parameter a_4, so we do not have other solutions.

Case 2: $a_2 = 2$. By direct computation, we get that the second largest eigenvalue of the graph $C(1, 2, 1, 1, 5)$ is greater than 1. Hence, $a_5 \leq 4$, and for $a_5 = 4$ the solution is $(a_1, a_2, a_3, a_4, a_5) = (1, 2, k, 1, 4)$. We now distinguish two more subcases depending on a_5. If $a_5 = 3$, we get $(a_1, a_2, a_3, a_4, a_5) = (1, 2, k, 2, 3)$, while if $a_5 = 2$ we get $(2^3, k, 2)$, where $k \geq 1$.

Case 3: $a_2 = 1$. The second largest eigenvalue of the graph $C(1^8, 9)$ is greater than 1. Hence, we get that $a_5 \leq 8$ and that one solution is $(a_1, a_2, a_3, a_4, a_5)=(1^4, 8)$. By putting $a_5 = 7, 6, \ldots, 1$ into the corresponding determinant we get the following solutions: $(a_1, a_2, a_3, a_4, a_5) = (1^2, 2, 1, 6)$, $(1^2, 4, 1, 5)$, $(2, 1, k, 1, 4)$, $(2, 1, k, 2, 3)$, $(4, 1, 2, 1, 3)$, $(3, 1, 4, 1, 3)$, $(3, 1, 6, k, 2)$, $(4, 1, 4, k, 2)$, and $(6, 1, 3, k, 2)$ (in all cases $k \geq 1$).

The proof is complete. □

Theorem 3.5. *If $C(a_1, a_2, \ldots, a_6)$ is a connected NSG satisfying $\lambda_2 = 1$ then we have* $(a_1, a_2, a_3, a_4, a_5, a_6) = (1, 3, 1, 4, 1, 2)$, $(1, 3, 1, 2, 1, 3)$, $(1, 3, 1^3, 5)$, $(3, 2, 1, 2, 1, 2)$, $(2^2, 1, 4, 1, 2)$, $(2^2, 1, 2, 1, 3)$, $(2^2, 1^3, 5)$, $(1, 2, k, 6, 1, 2)$, $(1, 2, k, 4, 1, 3)$, $(1, 2, k, 3, 1, 5)$, $(2, 1, k, 6, 1, 2)$, $(6, 1^2, 5, 1, 2)$, $(4, 1, 2, 5, 1, 2)$, $(3, 1, 4, 5, 1, 2)$, $(1^3, 4, 1, 4)$, $(2, 1, k, 4, 1, 3)$, $(3, 1, 5, 4, 1, 2)$, $(4, 1, 3, 4, 1, 2)$, $(6, 1, 2, 4, 1, 2)$, $(1^3, 3, 1, 13)$, $(1^2, 2, 3, 1, 9)$, $(1^2, 4, 3, 1, 7)$, $(1^2, 8, 3, 1, 6)$, $(2, 1, k, 3, 1, 5)$, $(6, 1^2, 3, 1, 3)$, $(4, 1, 2, 3, 1, 3)$, $(3, 1, 4, 3, 1, 3)$, $(8, 1, 2, 3, 1, 2)$, $(1^2, 2^4)$, $(1^3, 2^2, 3)$, $(3, 1^2, 2, 1, 11)$, $(3, 1, 2^2, 1, 9)$, $(3, 1, 3, 2, 1, 7)$, $(3, 1, 4, 2, 1, 5)$, $(3, 1, 5, 2, 1, 3)$, $(4, 1^2, 2, 1, 7)$, $(4, 1, 2^2, 1, 5)$, $(4, 1, 3, 2, 1, 3)$, $(6, 1^2, 2, 1, 5)$, $(6, 1, 2^2, 1, 3)$, $(10, 1^2, 2, 1, 4)$, $(10, 1, 2^2, 1, 2)$, $(1^2, 12, 1, 3, 2)$, $(1^2, 8, 1, 3^2)$, $(1^2, 6, 1, 3, 5)$, $(1^2, 5, 1, 3, 9)$, $(3, 1, 2, 1, 2^2)$, $(12, 1, 2, 1^2, 2)$, $(8, 1, 2, 1^2, 3)$, $(3, 1, 5, 1^2, 5)$, $(4, 1, 3, 1^2, 5)$, $(6, 1, 2, 1^2, 5)$, $(38, 1^4, 6)$, $(22, 1^4, 7)$, $(14, 1^4, 9)$, $(5, 1, 2, 1^2, 9)$, $(10, 1^4, 13)$, $(7, 1^4, 37)$, *or* $(8, 1^4, 21)$, *where* $k \geq 1$.

Proof. Due to Lemma 3.3 (ii) we have that $a_2 \leq 3$, $a_4 \leq 6$, $a_5 \leq 6$. We now distinguish three cases depending on a_2.

Case 1: $a_2 = 3$. Here we get: $a_1 = a_3 = a_5 = 1$, $a_4 \leq 4$, $a_6 \leq 5$. By distinguishing four subcases depending on a_4 we get the following solutions $(a_1, a_2, a_3, a_4, a_5, a_6) = (1, 3, 1, 4, 1, 2)$, $(1, 3, 1, 2, 1, 3)$, and $(1, 3, 1^3, 5)$.

Case 2: $a_2 = 2$. Here, we get $a_1 \leq 3$, $a_4 \leq 6$, $a_5 \leq 2$. We now distinguish two subcases depending on a_5. By considering the determinant of (2.1) where the parameters a_5 and a_2 are fixed, and discussing the remaining parameters we conclude the following: if $a_5 = 2$ we have no solutions; if $a_5 = 1$ we have the solutions $(a_1, a_2, a_3, a_4, a_5, a_6) = (3, 1, 1, 2, 1, 2)$, $(2^2, 1, 4, 1, 2), (2^2, 1, 2, 1, 3)$, $(2^2, 1^3, 5)$, $(1, 2, k, 6, 1, 2)$, $(1, 2, k, 4, 1, 3)$, and $(1, 2, k, 3, 1, 5)$.

Case 3: $a_2 = 1$. Due to Lemma 3.3 (ii) we have $a_4 \leq 6$ and $a_5 \leq 6$. After distinguishing six cases depending on a_4 and (if necessary) some subcases, similarly as in previous cases, we complete the above list.

The proof is complete. □

Theorem 3.6. *If $C(a_1, a_2, \ldots, a_7)$ is a connected NSG satisfying $\lambda_2 = 1$ then we have $(a_1, a_2, a_3, a_4, a_5, a_6, a_7) = (2, 1, k, 2, 1, l, 2)$, $(1, 2, k, 2, 1, l, 2)$, $(1, 2, k, 1, 2, l, 2)$, $(1^2, 4, 1^3, 3)$, $(1^2, 2, 1, 2, 1, 3)$, $(1^4, 4, 1, 3)$, $(6, 1^4, k, 2)$, $(4, 1, 2, 1^2, k, 2)$, $(3, 1, 4, 1^2, k, 2)$, $(2, 1, k, 1, 2, l, 2)$, $(1^4, 6, k, 2)$, $(1^2, 2, 1, 4, k, 2)$, or $(1^2, 4, 1, 3, k, 2)$, where $k, l \geq 1$.*

Proof. Due to Lemma 3.3 (iii) we have that $a_1 \leq 6$, $a_2 \leq 2$, $a_4 \leq 2$, $a_5 \leq 6$, and $a_7 \leq 3$. We distinguish two cases depending on a_4.

Case 1: $a_4 = 2$. This implies $a_1 \leq 2$, $a_2 \leq 2$, $a_5 = 1$, $a_7 = 2$. If $a_1 = 2$ we get $a_2 = a_5 = 1$, and $a_7 = 2$. By putting fixed values for a_1, a_2, a_4, a_5, and a_7 into (2.1), we get that the determinant is zero for any choice of parameters a_3, and a_6. Hence, we have the solution $(2, 1, k, 2, 1, l, 2)$, where $k, l \geq 1$. Similarly, if $a_1 = 1$ we get $a_2 \leq 2$. If $a_2 = 2$, by putting fixed values for a_1, a_2, a_4, a_5, and a_7 into (2.1), we get that the determinant is zero for any choice of parameters a_3, and a_6. Hence, we have the solutions $(1, 2, k, 2, 1, l, 2)$, where $k, l \geq 1$. Finally, for $a_2 = 1$, we get that the determinant is negative for any choice of non–fixed parameters a_3, and a_6.

Case 2: $a_4 = 1$. Due to Lemma 3.3 (iii) we have $a_1 \leq 6$, $a_2 \leq 2$, $a_5 \leq 6$, and $a_7 \leq 3$. By distinguishing two subcases depending on a_2 and some subcases we complete the above list.

The proof is complete. □

Theorem 3.7. *If $C(a_1, a_2, \ldots, a_8)$ is a connected NSG satisfying $\lambda_2 = 1$ then we have $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8) = (1, 2, k, 1^2, 4, 1, 2)$, $(1, 2, k, 1^2, 2, 1, 3)$, $(1, 2, k, 1^4, 5)$, $(1^3, 2, 1, 2, 1, 2)$, $(1^4, 2, 1, 2^2)$, $(2, 1, k, 1^2, 4, 1, 2)$, $(2, 1, k, 1^2, 2, 1, 3)$, $(2, 1, k, 1^4, 5)$, $(1^2, 4, 1^2, 5, 1, 2)$, $(1^2, 2, 1, 2, 5, 1, 2)$, $(1^4, 4, 5, 1, 2)$, $(1^4, 5, 4, 1, 2)$, $(1^2, 2, 1, 3, 4, 1, 2)$, $(1^2, 4, 1, 2, 4, 1, 2)$, $(1^4, 4, 3, 1, 3)$, $(1^2, 2, 1, 2, 3, 1, 3)$, $(1^2, 6, 1, 2, 3, 1, 2)$, $(1^2, 4, 1^2, 3, 1, 3)$, $(1^4, 5, 2, 1, 3)$, $(1^4, 4, 2, 1, 5)$, $(1^2, 2, 1, 3, 2, 1, 5)$, $(1^4, 3, 2, 1, 7)$, $(1^2, 8, 1, 2^2, 1, 2)$, $(1^2, 4, 1, 2^2, 1, 3)$, $(1^2, 2, 1, 2^2, 1, 5)$, $(1^4, 2^2, 1, 9)$, $(1^5, 2, 1, 11)$, $(1^2, 2, 1^2, 2, 1, 7)$, $(1^2, 4, 1^2, 2, 1, 5)$, $(1^2, 8, 1^2, 2, 1, 4)$, $(1^4, 5, 1^2, 8)$, $(1^2, 2, 1, 3, 1^2, 5)$, $(1^2, 3, 1, 2, 1^2, 9)$, $(1^2, 4, 1, 2, 1^2, 5)$, $(1^2, 6, 1, 2, 1^2, 3)$, $(1^2, 10, 1, 2, 1^2, 2)$, $(1^2, 5, 1^4, 37)$, $(1^2, 6, 1^4, 21)$, $(1^2, 8, 1^4, 13)$, $(1^2, 12, 1^4, 9)$, $(1^2, 20, 1^4, 7)$ or $(1^2, 36, 1^4, 6)$, where $k \geq 1$.*

Proof. Due to Lemma 3.3 (iv) we have that $a_1 \leq 3$, $a_2 \leq 2$, $a_4 \leq 2$, $a_5 \leq 5$, $a_6 \leq 5$, and $a_7 \leq 3$. Similarly as in the previous theorems we get the first solutions by choosing $a_2 = 2$, the next solution by choosing $a_2 = 1$, and $a_4 = 2$. The remaining solutions are obtained for $a_2 = 1$, $a_4 = 1$, and by distinguishing the cases depending on a_7. □

Finally, we have the following result.

Theorem 3.8. *If $C(a_1, a_2, \ldots, a_9)$ is a connected NSG satisfying $\lambda_2 = 1$ then we have $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9) = (1^2, 4, 1^5, 2)$, $(1^2, 2, 1, 2, 1^3, 2)$, or $(1^4, 4, 1^3, 2)$.*

Proof. Due to Lemma 3.3 (v) we have $a_1 = a_2 = a_4 = a_6 = a_7 = 1$, $a_3 \leq 4$, $a_5 \leq 4$, and $a_9 = 2$. By putting fixed values for a_1, a_2, a_4, a_6, a_7, and a_9 into (2.1) we get $\det(I - D) = 2(a_3a_5 - 4)$. Hence, the determinant is equal to zero when $(a_3, a_5) = (4, 1)$, or $(a_3, a_5) = (2, 2)$, or $(a_3, a_5) = (1, 4)$, and the proof follows. □

Collecting the above results we get the following theorem.

Theorem 3.9. *Let G be a connected NSG satisfying $\lambda_2 = 1$. Then it is some of the graphs presented in Table 1. Those graphs are represented by the parameters $a_1, a_2, \ldots, a_n$, while k and l stand for any positive integers.*

The infinite families of NSGs satisfying $\lambda_2 = 1$ deserve a special attention. There are many results concerning the graphs with maximal index (for example, see [9] and its references). As we pointed out, each graph with maximal index and fixed order and size is an NSG. So, the results presented in [10] and here can be considered from that point of view. Namely, all graphs with maximal index (having fixed order and size) whose second largest eigenvalue does not exceed 1 can be identified by considering Theorem 4.8 (from [10]), and Theorem 3.9 (from here).

We conclude the paper by the list of the obtained NSGs. The graphs are ordered by n and lexicographically.

G	a_1	a_2	a_3	a_4	a_5	a_6	G	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
1.	3	1	8				81.	4	1	3	2	1	3			
2.	1	2	7				82.	6	1	1	2	1	5			
3.	4	1	6				83.	6	1	3	2	1	3			
4.	6	1	5				84.	10	1	1	2	1	4			
5.	1	3	4				85.	10	1	2	2	1	2			
6.	2	2	4				86.	1	1	12	1	3	2			
7.	1	4	3				87.	1	1	8	1	3	3			
8.	2	3	3				88.	1	1	6	1	3	5			
9.	14	1	3	2			89.	1	1	5	1	3	9			
10.	10	1	3	3			90.	3	1	2	1	2	2			
11.	8	1	3	5			91.	12	1	2	1	1	2			
12.	7	1	3	9			92.	8	1	2	1	1	3			
13.	5	1	4	2			93.	3	1	5	1	1	5			
14.	4	2	2	2			94.	4	1	3	1	1	5			
15.	3	2	2	3			95.	6	1	2	1	1	5			
16.	1	8	1	2			96.	38	1	1	1	1	6			
17.	2	7	1	2			97.	22	1	1	1	1	7			
18.	1	6	1	3			98.	14	1	1	1	1	9			
19.	1	5	1	5			99.	5	1	2	1	1	9			
20.	2	5	1	3			100.	10	1	1	1	1	13			
21.	2	4	1	5			101.	7	1	1	1	1	37			
22.	3	4	1	4			102.	8	1	1	1	1	21			
23.	3	3	1	13			103.	2	1	k	2	1	l	2		
24.	4	3	1	9			104.	1	2	k	2	1	l	2		
25.	6	3	1	7			105.	1	2	k	1	2	l	2		
26.	10	3	1	6			106.	1	1	4	1	1	1	3		
27.	1	4	1	k	2		107.	1	1	2	1	2	1	3		
28.	1	3	2	k	2		108.	1	1	1	1	4	1	3		
29.	2	3	1	k	2		109.	6	1	1	1	1	1	2		
30.	1	2	k	1	4		110.	4	1	2	1	1	k	2		
31.	1	2	k	2	3		111.	3	1	4	1	1	k	2		
32.	2	2	2	k	2		112.	2	1	k	1	2	l	2		
33.	1	1	1	1	8		113.	1	1	1	1	6	k	2		
34.	1	1	2	1	6		114.	1	1	2	1	4	k	2		
35.	1	1	4	1	5		115.	1	1	4	1	3	k	2		
36.	2	1	k	1	4		116.	1	2	k	1	1	4	1	2	
37.	2	1	k	2	3		117.	1	2	k	1	1	2	1	3	
38.	6	1	1	1	3		118.	1	2	k	1	1	1	1	5	
39.	4	1	2	1	3		119.	1	1	1	2	1	2	1	2	
40.	3	1	4	1	3		120.	1	1	1	1	2	1	2	2	
41.	3	1	6	k	2		121.	2	1	k	1	1	4	1	2	
42.	4	1	4	k	2		122.	2	1	k	1	1	2	1	3	
43.	6	1	3	k	2		123.	2	1	k	1	1	1	1	5	
44.	1	3	1	4	1	2	124.	1	1	4	1	1	5	1	2	
45.	1	3	1	2	1	3	125.	1	1	2	1	2	5	1	2	
46.	1	3	1	1	1	5	126.	1	1	1	1	4	5	1	2	
47.	3	2	1	2	1	2	127.	1	1	1	1	5	4	1	2	
48.	2	2	1	4	1	2	128.	1	1	2	1	3	4	1	2	
49.	2	2	1	2	1	3	129.	1	1	4	1	2	4	1	2	
50.	2	2	1	1	1	5	130.	1	1	1	1	4	3	1	3	
51.	1	2	k	6	1	2	131.	1	1	2	1	2	3	1	3	
52.	1	2	k	4	1	3	132.	1	1	6	1	2	3	1	2	
53.	1	2	k	3	1	5	133.	1	1	4	1	1	3	1	3	
54.	2	1	k	6	1	2	134.	1	1	1	1	5	2	1	3	
55.	6	1	1	5	1	2	135.	1	1	1	1	4	2	1	5	
56.	4	1	2	5	1	2	136.	1	1	2	1	3	2	1	5	
57.	3	1	4	5	1	2	137.	1	1	1	1	3	2	1	7	
58.	1	1	1	4	1	4	138.	1	1	8	1	2	2	1	2	
59.	2	1	k	4	1	3	139.	1	1	4	1	2	2	1	3	
60.	3	1	5	4	1	2	140.	1	1	2	1	2	2	1	5	
61.	4	1	3	4	1	2	141.	1	1	1	1	2	2	1	9	
62.	6	1	2	4	1	2	142.	1	1	1	1	1	2	1	11	
63.	1	1	1	3	1	13	143.	1	1	2	1	1	2	1	7	
64.	1	1	2	3	1	9	144.	1	1	4	1	1	2	1	5	
65.	1	1	4	3	1	7	145.	1	1	8	1	1	2	1	4	
66.	1	1	8	3	1	6	146.	1	1	1	1	5	1	1	8	
67.	2	1	k	3	1	5	147.	1	1	2	1	3	1	1	5	
68.	6	1	1	3	1	3	148.	1	1	3	1	2	1	1	9	
69.	4	1	2	3	1	3	149.	1	1	4	1	2	1	1	5	
70.	3	1	4	3	1	3	150.	1	1	6	1	2	1	1	3	
71.	8	1	2	3	1	2	151.	1	1	10	1	2	1	1	2	
72.	1	1	2	2	2	2	152.	1	1	5	1	1	1	1	37	
73.	1	1	1	2	2	3	153.	1	1	6	1	1	1	1	21	
74.	3	1	1	2	1	11	154.	1	1	8	1	1	1	1	13	
75.	3	1	2	2	1	9	155.	1	1	12	1	1	1	1	9	
76.	3	1	3	2	1	7	156.	1	1	20	1	1	1	1	7	
77.	3	1	4	2	1	5	157.	1	1	36	1	1	1	1	6	
78.	3	1	5	2	1	3	158.	1	1	4	1	1	1	1	1	2
79.	4	1	1	2	1	7	159.	1	1	2	1	2	1	1	1	2
80.	4	1	2	2	1	5	160.	1	1	1	1	4	1	1	1	2

Table 1: NSGs with $\lambda_2 = 1$.

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