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A note on a walk-based inequality for the index of a signed graph

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Abstract: We derive an inequality that includes the largest eigenvalue of the adjacency matrix and walks of an arbitrary length of a signed graph. We also consider certain particular cases.

Keywords: Signed graph; walk; adjacency matrix; index; upper bound

MSC: 05C22; 05C50

1 Introduction

A *signed graph* $\dot{G}$ is a pair (G, σ), where $G = (V, E)$ is an unsigned graph, called the *underlying graph*, and $\sigma\colon E \longrightarrow \{-1, +1\}$ is the *sign function*. We denote the number of vertices of a signed graph by n. The edge set of a signed graph is composed of subsets of positive and negative edges. Throughout the paper we interpret an unsigned graph as a signed graph with all the edges being positive.

The $n \times n$ adjacency matrix $A_{\dot{G}}$ of $\dot{G}$ is obtained from the standard $(0, 1)$-adjacency matrix of G by reversing the sign of all 1s which correspond to negative edges. The largest eigenvalue of $A_{\dot{G}}$ is called the *index* of $\dot{G}$ and denoted by λ_1. A detailed introduction to spectra of signed graphs can be found in [3].

Spectra of signed graph have received a great deal of attention in the recent years. In particular, some upper bounds for λ_1 appeared in our previous works [1, 2]. In this note we generalize the result of [2] concerning an upper bound for λ_1 in terms of certain standard invariants. Additional terminology and notation are given in Section 2. Our contribution and some consequences are given in Section 3.

2 Terminology and notation

If the vertices i and j are adjacent, then we write $i \sim j$. In particular, the existence of a positive (resp. negative) edge between these vertices is designated by $i \overset{+}{\sim} j$ (resp. $i \overset{-}{\sim} j$). We use d_i to denote the degree of a vertex $i \in V(\dot{G})$; in particular, we write d_i^+ and d_i^- for the positive and negative vertex degree (i.e., the number of positive and negative edges incident with i), respectively. For (not necessary distinct) vertices i and j, we use c_{ij}^{++} to denote the number of their common neighbours joined to both of them by a positive edge, c_{ij}^{+} to denote the the number of their common neighbours joined to i by a positive edge and to j by any edge. We also use the similar notation for all the remaining possibilities.

The definition of a walk in a signed graph does not deviate from the same definition in the case of graphs. So, a *walk* is a sequence of alternate vertices and edges such that consecutive vertices are incident with the corresponding edge. A walk in a signed graph is *positive* if the number of its negative edges (counted with

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their multiplicity if there are repeated edges) is not odd; otherwise, it is negative. In the same way we decide whether a cycle in a signed graph is positive or negative. We use $w_r^+(i,j)$ and $w_r^+(i)$ to denote the number of positive walks of length k starting at i and terminating at j and the number of positive walks of length k starting at i, respectively, and similarly for the numbers of negative ones.

3 Results

Our main result reads as follows.

Theorem 3.1. *For the index λ_1 of signed graph $\dot{G}$,*

$$\lambda_1\left(n_i^+ + n_i^- + \lambda_1^{r-1}\right) \leq (n_i^+ + n_i^-)d_i + \sum_{j=1}^{n}(w^+ + w^-)d_j - 2\Big(\sum_{j\,:\,w^+\neq 0}(c_{ij}^{+-} + c_{ij}^{-+}) + \sum_{j\,:\,w^-\neq 0}(c_{ij}^{++} + c_{ij}^{--})\Big),$$

where i is a vertex that corresponds to the largest absolute value of the coordinates of an eigenvector afforded by λ_i, r ($r \geq 2$) is an integer, $w^+ = w_{r-1}^+(i,j)$, $w^- = w_{r-1}^-(i,j)$ and n_i^+ (resp. n_i^-) is the number of vertices j such that $w^+ \neq 0$ (resp. $w^- \neq 0$).

Proof. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)^\intercal$ be an eigenvector associated with λ_1 and let x_i be the coordinate that is largest in absolute value. Without loss of generality, we may assume that $x_i = 1$. Considering the ith and the jth equality of $\lambda_1\mathbf{x} = A_{\dot{G}}\mathbf{x}$, we get

$$\lambda_1 = \sum_{k\overset{+}{\sim}i} x_k - \sum_{k\overset{-}{\sim}i} x_k, \tag{1}$$

and

$$\lambda_1 x_j = \sum_{k\overset{+}{\sim}j} x_k - \sum_{k\overset{-}{\sim}j} x_k. \tag{2}$$

By multiplying the equality (2) by $w^+ = w_{r-1}^+(i,j)$ and adding to (1), we get

$$\begin{aligned}
\lambda_1(1 + w^+x_j) &= \sum_{k\overset{+}{\sim}i} x_k - \sum_{k\overset{-}{\sim}i} x_k + w^+\Big(\sum_{k\overset{+}{\sim}j} x_k - \sum_{k\overset{-}{\sim}j} x_k\Big)\\
&= (1 + w^+)\Big(\sum_{\substack{k\overset{+}{\sim}i\\ k\overset{+}{\sim}j}} x_k - \sum_{\substack{k\overset{-}{\sim}i\\ k\overset{-}{\sim}j}} x_k\Big) + (1 - w^+)\Big(\sum_{\substack{k\overset{+}{\sim}i\\ k\overset{-}{\sim}j}} x_k - \sum_{\substack{k\overset{-}{\sim}i\\ k\overset{+}{\sim}j}} x_k\Big)\\
&\quad + \sum_{\substack{k\overset{+}{\sim}i\\ k\nsim j}} x_k - \sum_{\substack{k\overset{-}{\sim}i\\ k\nsim j}} x_k + w^+\Big(\sum_{\substack{k\nsim i\\ k\overset{+}{\sim}j}} x_k - \sum_{\substack{k\nsim i\\ k\overset{-}{\sim}j}} x_k\Big)\\
&\leq (1 + w^+)(c_{ij}^{++} + c_{ij}^{--}) + |1 - w^+|(c_{ij}^{+-} + c_{ij}^{-+}) + d_i^+ - c_{ij}^+ + d_i^- - c_{ij}^- + w^+(d_j^+ - c_{ji}^+ + d_j^- - c_{ji}^-)\\
&= d_i + w^+d_j - (1 + w^+ - |1 - w^+|)(c_{ij}^{+-} + c_{ij}^{-+}).
\end{aligned}$$

Observe that, for $w^+ \neq 0$, the previous inequality reduces to

$$\lambda_1(1 + w^+x_j) \leq d_i + w^+d_j - 2(c_{ij}^{+-} + c_{ij}^{-+}).$$

Taking the summation over all j such that $w^+ \neq 0$, we get

$$\lambda_1\Big(n_i^+ + \sum_{j\,:\,w^+\neq 0} w^+x_j\Big) \leq n_i d_i + \sum_{j\,:\,w^+\neq 0}\big(w^+d_j - 2(c_{ij}^{+-} + c_{ij}^{-+})\big). \tag{3}$$

Similarly, by multiplying the equality (2) by $w^- = w_{r-1}^-(i,j)$ and subtracting it from (1), we get

$$\lambda_1(1 - w^-x_j) \leq d_i + w^-d_j - (1 + w^- - |1 - w^+|)(c_{ij}^{++} + c_{ij}^{--}),$$

which, after taking the summation over all j such that $w^- \neq 0$, leads to

$$\lambda_1\Big(n_i^- - \sum_{j\,:\,w^-\neq 0}^{n} w^- x_j\Big) \le n_i^- d_i + \sum_{j\,:\,w^-\neq 0}^{n}\big(w^- d_j - 2(c_{ij}^{++} + c_{ij}^{--})\big). \tag{4}$$

Since

$$\lambda^{r-1} = \lambda^{r-1} x_i = \sum_{j=1}^{n}(w^+ - w^-)x_j = \sum_{j\,:\,w^+\neq 0} x_j - \sum_{j\,:\,w^-\neq 0} x_j,$$

by summing (3) and (4), we obtain

$$\lambda_1\big(n_i^+ + n_i^- + \lambda_1^{r-1}\big) \le (n_i^+ + n_i^-)d_i + \sum_{j=1}^{n}(w^+ + w^-)d_j - 2\Big(\sum_{j\,:\,w^+\neq 0}(c_{ij}^{+-} + c_{ij}^{-+}) + \sum_{j\,:\,w^-\neq 0}(c_{ij}^{++} + c_{ij}^{--})\Big),$$

which completes the proof. □

The Laplacian matrix $L_{\dot{G}}$ is defined as $L_{\dot{G}} = D_{\dot{G}} - A_{\dot{G}}$, where $D_{\dot{G}}$ is the diagonal matrix of vertex degrees. Observe that the counterparts to (1) and (2) in the case of the Laplacian matrix $L_{\dot{G}}$ are given by $\mu_1 = d_i + \sum_{k\overset{+}{\sim}i} x_k - \sum_{k\overset{-}{\sim}i} x_k$ and $\mu_1 x_j = d_j x_j + \sum_{k\overset{+}{\sim}i} x_k - \sum_{k\overset{-}{\sim}i} x_k$ (μ_1 being the largest eigenvalue of $L_{\dot{G}}$). Now, with slight modifications in the previous proof, we get the following.

Theorem 3.2. *For the Laplacian index μ_1 of signed graph $\dot{G}$,*

$$\mu_1\big(n_i^+ + n_i^- + \mu_1^{r-1}\big) \le 2\Big((n_i^+ + n_i^-)d_i + \sum_{j=1}^{n}(w^+ + w^-)d_j - \sum_{j\,:\,w^+\neq 0}(c_{ij}^{+-} + c_{ij}^{-+}) + \sum_{j\,:\,w^-\neq 0}(c_{ij}^{++} + c_{ij}^{--})\Big),$$

with the notations of Theorem 3.1.

For $r = 2$, we have $n_i^+ = w^+ = d_i^+$, $n_i^- = w^- = d_i^-$, while $\sum_{j\,:\,w^+\neq 0}(c_{ij}^{+-} + c_{ij}^{-+}) + \sum_{j\,:\,w^-\neq 0}(c_{ij}^{++} + c_{ij}^{--}) = 2T_i^-$, i.e., this is twice the sum of negative triangles passing through i. Thus Theorem 3.1 gives $\lambda_1(d_i + \lambda_1) \le d_i^2 + d_i m_i - 4T_i^-$, where m_i is the average degree of the neighbours of i. This quadratic equation leads to

$$\lambda_1^2 \le \max_{1\le i\le n}\Big\{\frac{1}{2}\Big(\sqrt{5d_i^2 + 4(d_i m_i - 4T_i^-)} - d_i\Big)\Big\},$$

the upper bound obtained in [2].

For $r = 3$, we get

$$\lambda_1\big(n_i^+ + n_i^- + \lambda_1^2\big) \le (n_i^+ + n_i^-)d_i + \sum_{j\,:\,w^+\neq 0}(w^+ d_j - 2w^-) + \sum_{j\,:\,w^-\neq 0}(w^- d_j - 2w^+),$$

as $c_{ij}^{+-} + c_{ij}^{-+} = w^-$, $c_{ij}^{++} + c_{ij}^{--} = w^+$. In particular case of graphs, the latter inequality reduces to

$$\lambda_1\big(d_2(i) + 1 + \lambda_1^2\big) \le (d_2(i) + 1)d_i + w_3(i),$$

where $d_2(i)$ denotes the number of vertices at distance 2 from i (and then $n_i^+ = d_2(i) + 1$) and $w_3(i)$ denotes the number of walks of length 3 starting at i.

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