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Star complements for ±2 in signed graphs

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Abstract: In this article, we investigate connected signed graphs which have a connected star complement for both −2 and 2 (i.e. simultaneously for the two eigenvalues), where −2 (resp. 2) is the least (largest) eigenvalue of the adjacency matrix of a signed graph under consideration. We determine all such star complements and their maximal extensions (again, relative to both eigenvalues). As an application, we provide a new proof of the result which identifies all signed graphs that have no eigenvalues other than −2 and 2.

Keywords: signed graph eigenvalue, star complement, maximal extension, signed line graph

MSC 2020: 05C50, 05C22

1 Introduction

A signed graph $\dot{G}$ is a pair (G, σ), where G is an (unsigned) graph (V, E), called the underlying graph, and $\sigma : E \longrightarrow \{1, -1\}$ is the *sign function* or *signature*. The order of $\dot{G}$, denoted by n, is the number of its vertices. The edge set of $\dot{G}$ consists of positive and negative edges determined by σ. We interpret a graph as a signed graph whose signature assigns 1 to each edge.

The *adjacency matrix* $A_{\dot{G}}$ of $\dot{G}$ is the $(0, 1, -1)$-matrix of size $n \times n$ obtained from the adjacency matrix of its underlying graph by reversing the sign of each entry which corresponds to a negative edge. The *eigenvalues* of $\dot{G}$ are the eigenvalues of its adjacency matrix, and they form the *spectrum* of $\dot{G}$.

We consider connected star complements (this notion is defined in the next section) for −2 as the least eigenvalue which are simultaneously star complements for 2 as the largest eigenvalue. We determine all such star complements and their maximal extensions. As an application we provide a new proof of the result from [1–3] which determines all signed graphs with exactly two distinct eigenvalues: −2 and 2. This result is interesting because, according to [2], a signed graph whose spectrum lies in $[-2, 2]$ is an induced subgraph of a signed graph with eigenvalues −2 and 2.

Section 2 is preparatory. Our contribution is reported in Sections 3 and 4.

2 Preliminaries

We use λ and Λ to denote the least and the largest eigenvalue of a signed graph, respectively.

For a subset $U \subseteq V(\dot{G})$, let $\dot{G}^U$ denote the signed graph obtained by switching the sign of the edges between U and $V(\dot{G})\backslash U$. The signed graphs $\dot{G}$ and $\dot{G}^U$ are said to be *switching equivalent*.

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A cycle in a signed graph is said to be *positive* if the product of its edge signs is 1. Otherwise, it is said to be *negative*. A signed graph is *balanced* if it does not contain a negative cycle, for otherwise it is *unbalanced*.

In the reminder of this section, we introduce the concepts of star complements and signed line graphs. The first concept is a natural extension of the concept developed in the framework of unsigned graphs and can also be found in [4]. The second concept, which can be found in [5,6], is tailored for the spectral theory of signed graphs and differs in sign from that of [7].

2.1 Star complements in signed graphs

Let $\dot{G}$ be a signed graph. A subset X of $V(\dot{G})$ is said to be a *star set* for the eigenvalue ν of multiplicity k if $|X| = k$ and ν is not an eigenvalue of $\dot{G} - X$. The induced subgraph $\dot{G} - X$ is the corresponding *star complement* for ν.

The following result is a “signed” generalization of the well-known Reconstruction theorem [8, Theorem 5.1.7].

Theorem 2.1. *Let $\dot{G}$ be a signed graph with*

$$A_{\dot{G}} = \begin{pmatrix} A_X & B^{\intercal} \\ B & C \end{pmatrix},$$

where A_X is the adjacency matrix of the subgraph induced by $X \subset V(\dot{G})$. Then X is a star set for ν if and only if ν is not an eigenvalue of C and

$$\nu I - A_X = B^{\intercal}(\nu I - C)^{-1}B.$$

If X is a star set for the eigenvalue ν, $|X| = k$, and $\dot{G}$ has order n, then we define a bilinear form on $\mathbb{R}^{n-k}$ as follows:

$$\langle \mathbf{x}, \mathbf{y} \rangle := \mathbf{x}^{\intercal}(\nu I - C)^{-1}\mathbf{y}.$$

Also, if $u \in X$ then we write $\mathbf{b}_u$ for the characteristic vector of the neighbourhood of u in the corresponding star complement; this vector carries the signs of the corresponding edges.

From Theorem 2.1 we have:

$$\langle \mathbf{b}_u, \mathbf{b}_v \rangle = \begin{cases} \nu & \text{if } u = v, \\ 0 & \text{if } u \nsim v,\ u \neq v, \\ 1 & \text{if } u \overset{-}{\sim} v, \\ -1 & \text{if } u \overset{+}{\sim} v. \end{cases}$$

We now introduce some additional terminology and notation related to star complements. Given a signed graph $\dot{H}$, we fix a vertex $u \notin V(\dot{H})$ and a subset $U \subseteq V(\dot{H})$, and we let $\dot{H}(U)$ be a signed graph obtained by inserting an edge between u and every vertex of U and taking a particular signature for these edges. If ν is an eigenvalue of $\dot{H}(U)$ but is not an eigenvalue of $\dot{H}$, we say that u, U and $\dot{H}(U)$ are a *good vertex*, a *good set*, and a *good extension* for ν, respectively. Moreover, we say that the vertices $u, v \notin V(\dot{H})$ are *compatible* for ν (and the same is said for the corresponding sets U, V) if u, v are good and $\langle \mathbf{b}_u, \mathbf{b}_v \rangle \in \{0, 1, -1\}$. Theorem 2.1 tells us that a vertex set X in which all vertices are good and compatible in pairs affords a good extension $\dot{G}$ in which X and $\dot{H}$ are a star set and a star complement for ν. If such a signed graph $\dot{G}$ is not a proper induced subgraph of some other signed graph with the same star complement for the same eigenvalue, then $\dot{G}$ is referred to as a *maximal* signed graph with star complement $\dot{H}$ for ν or a *maximal extension* of $\dot{H}$ for ν.

2.2 Signed line graphs

Given a signed graph $\dot{G}$, we introduce the vertex-edge orientation $\eta : V(\dot{G}) \times E(\dot{G}) \longrightarrow \{0, 1, -1\}$ which obeys (a) $\eta(i, jk) = 0$ if $i \notin \{j, k\}$, (b) $\eta(i, ij) = 1$ or $\eta(i, ij) = -1$, and (c) $\eta(i, ij)\eta(j, ij) = -\sigma(ij)$. Since each edge receives two orientations, η is also referred to as a *bi-orientation*. A *bi-directed graph* $\dot{G}_\eta = (\dot{G}, \eta)$ consists of $\dot{G}$ together with the bi-orientation η.

The (vertex-edge) *incidence matrix* B_η is the matrix whose rows and columns are indexed by the vertices and the edges of $\dot{G}$, respectively, such that its (i, e)-entry is equal to $\eta(i, e)$. We have

$$B_\eta^\intercal B_\eta = 2I + A_{\mathcal{L}(\dot{G}_\eta)},$$

where $\mathcal{L}(\dot{G}_\eta)$ is taken to be the *signed line graph* of $\dot{G}_\eta$. Note that $\mathcal{L}(\dot{G}_\eta)$ depends on the bi-orientation η, but a different bi-orientation results in a switching equivalent signed graph. Therefore, we have a signed line graph of $\dot{G}$ which is unique up to switching, and we denote it by $\mathcal{L}(\dot{G})$. In this context, $\dot{G}$ is called a *signed root graph* (of $\mathcal{L}(\dot{G})$).

Since $B_\eta^\intercal B_\eta$ is positive semidefinite, we have $\lambda(\mathcal{L}(\dot{G})) \geq -2$. A signed graph that is not a signed line graph, but whose least eigenvalue is ≥ -2, is called an *exceptional* signed graph.

In a signed graph, two parallel edges (located between the same pair of vertices) form a cycle of length 2 called a *digon*. A digon is *positive* if its edges have the same sign, and *negative* if they differ in sign. In particular, the existence of a positive digon in $\dot{G}$ implies the existence of parallel edges in $\mathcal{L}(\dot{G})$. Also, a negative digon produces non-adjacent vertices in $\mathcal{L}(\dot{G})$. A signed graph which allows parallel edges if and only if they form negative digons is called a *simply signed graph*. Accordingly, $\mathcal{L}(\dot{G})$ has no multiple edges if and only if $\dot{G}$ is a simply signed graph.

Moreover, given an unsigned graph G, we define the *signed doubled graph* $\ddot{G}$ as the signed graph obtained by inserting a parallel negative edge between every pair of adjacent vertices of G. We know from [6] that if G is regular of order n and vertex degree r, then the spectrum of $\mathcal{L}(\ddot{G})$ is $[2(r-1)^n, (-2)^{n(r-1)}]$, where in the exponential notation, the exponent stands for the eigenvalue multiplicity.

The following result is taken from [9]. Recall that a connected signed graph whose order is equal to its number of edges is called *unicyclic*.

Lemma 2.2. *A connected signed graph $\dot{G}$ with $\lambda(\dot{G}) > -2$ belongs to one of the following classes*:
(*i*) *Signed line graphs of trees*.
(*ii*) *Signed line graphs of unbalanced unicyclic simply signed graphs*.
(*iii*) *Exceptional signed graphs whose least eigenvalue is* >-2.

We know from [9,10] that the signed graphs of Lemma 2.2(iii) have at most eight vertices.

3 Star complements for ±2

We start by determining the potential star complements for ±2 in the role of the least and the largest eigenvalue.

Theorem 3.1. *The spectrum of a connected signed graph $\dot{G}$ lies in* $(-2, 2)$ *if and only if $\dot{G}$ is switching equivalent to an induced subgraph of* (*at least*) *one of the signed graphs illustrated in* Figure 1.

Proof. First, we confirm by inspection that the spectrum of each of the first ten signed graphs in Figure 1 lies in $(-2, 2)$, and the same holds for their induced subgraphs by the eigenvalue interlacing argument. For the remaining two (the negative quadrangle with two hanging paths and the negative even cycle), we observe that both are signed line graphs and in both cases the corresponding root is an unbalanced unicyclic simply

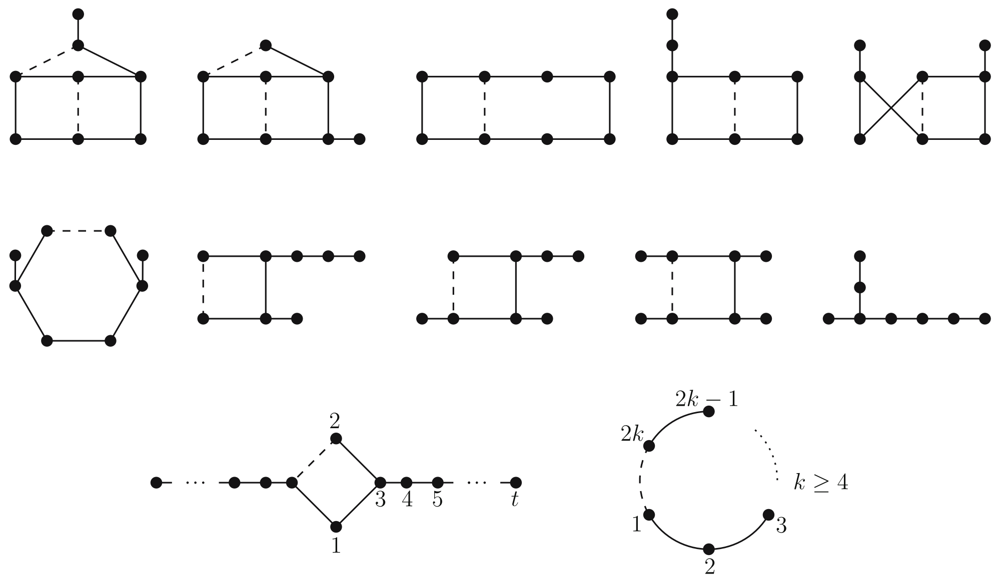

Figure 1: Connected signed graphs whose spectrum lies in $(-2, 2)$. Here and in subsequent figures, dashed lines represent negative edges.

signed graph. Therefore, their least eigenvalues are greater than -2 by Lemma 2.2(ii). Moreover, since both are bipartite, their spectra are symmetric about 0, and we are done.

Conversely, let $\dot{G}$ be a connected signed graph whose spectrum lies in $(-2, 2)$. First, $\dot{G}$ does not contain an odd cycle as an induced subgraph, since depending on the signature for such a cycle we have either $\lambda = -2$ or $\Lambda = 2$ (where λ and Λ are defined at the beginning of Section 2). Next, $\dot{G}$ cannot contain a positive even cycle as an induced subgraph, for otherwise both $|\lambda(\dot{G})|$ and $\Lambda(\dot{G})$ are at least 2. Hence, all induced cycles in $\dot{G}$ are negative and of even length. We proceed with more restrictions:

(i) $\dot{G}$ cannot contain an induced negative even cycle with at least eight vertices and a hanging edge. (Otherwise, $\dot{G}$ would contain the Smith graph obtained by attaching a hanging edge at the central vertex of a 7-vertex path, with $\Lambda = -\lambda = 2$.)

(ii) $\dot{G}$ cannot contain an induced signed graph in which two negative even cycles share more than one edge. (Observe that such an induced graph necessarily contains a negative odd cycle or a positive even cycle as an induced subgraph.)

By examining connected signed graphs with negative even cycles of length at most six and taking into account the restrictions of (i) and (ii), we arrive at the desired solutions. □

We denote the order of a star complement by t and proceed by determining maximal extensions of the candidates for star complements with $t \geq 9$. According to the previous theorem, there are five types of possible star complements for -2 and 2: (a) the negative quadrangle with two paths attached at its non-adjacent vertices, (b) the negative quadrangle with a single path attached at one of its vertices, (c) a path with an edge attached at the vertex labelled 2, (d) a path, and (e) an even negative cycle. We demonstrate two methods, one in case (a) and the other in case (c), and then comment on the remaining cases.

We start with the star complement (a) for both ±2.

Proposition 3.2. *For $t \geq 9$, $\mathcal{L}(\ddot{C}_t)$ is the unique maximal extension of the signed graph (a) for ±2.*

Proof. Since $t \geq 9$, the maximal extension, say $\dot{G}$, is a signed line graph. We observe next that the star complement is also a signed line graph of a $(t-1)$-vertex path with a negative digon; we denote this signed root graph by $\dot{H}$. Therefore, $\dot{G}$ is obtained by inserting the maximum number of edges in $\dot{H}$ and then taking the signed line graph of the resulting root graph. Note that the edges between non-adjacent vertices of $\dot{H}$ are disallowed since their existence produces positive triangles in $\dot{G}$ which cannot exist because the largest eigenvalue of a positive triangle with an attached edge is greater than 2. The remaining edges are the two edges (one positive and the other negative) between the pendant vertices of $\dot{H}$ and the edges that form negative digons. By inserting all of them we obtain the signed graph of Figure 2. This is $\mathcal{L}(\ddot{C}_t)$, and its spectrum is in $[2^t, (-2)^t]$ (from the formula in Section 2). Therefore, $\mathcal{L}(\ddot{C}_t)$ is indeed the maximal extension for -2 and 2. This extension is unique since it includes all possible good vertices. □

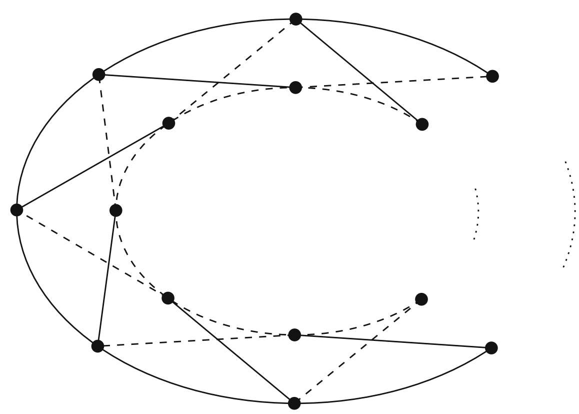

Figure 2: Signed line graph $\mathcal{L}(\ddot{C}_t)$.

We now investigate the graph (c) by a different method. We denote this graph by Z_t and observe that Z_t is a subgraph of the last but one signed graph of Figure 1, and so we can transfer the vertex labelling from this figure. If C is the adjacency matrix of Z_t, then the matrix $M = 4(2I - C)^{-1}$ is

$$M = \begin{pmatrix} t & t-2 & 2(t-2) & 2(t-3) & \cdots & 4 & 2 \\ t-2 & t & 2(t-2) & 2(t-3) & \cdots & 4 & 2 \\ 2(t-2) & 2(t-2) & 4(t-2) & 4(t-3) & \cdots & 8 & 4 \\ 2(t-3) & 2(t-3) & 4(t-3) & 4(t-3) & \cdots & 8 & 4 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 4 & 4 & 8 & 8 & \cdots & 8 & 4 \\ 2 & 2 & 4 & 4 & \cdots & 4 & 4 \end{pmatrix}, \tag{1}$$

which can be confirmed by direct computation. Thus for $i \geq 3$, the ith row of the upper triangle of the symmetric matrix M is $(4(k-i+1), 4(k-i), 4(k-i-1), \ldots, 4)$.

We first eliminate the possibility that u has at least four neighbours in Z_t. (We recall the reader that the vertex u and the corresponding set U are defined in Section 2.1.)

Lemma 3.3. *If $Z_t(U)$ is good, then $|U| \leq 3$.*

Proof. If $|U| \geq 5$, then (as odd triangles are disallowed) $Z_t(U)$ necessarily contains $K_{1,5}$ as an induced subgraph, with $\lambda < -2$. If $|U| = 4$, then by simple reasoning we conclude that, unless $t = 4$ or $t = 8$ and $U = \{1, 4, 6, 8\}$, $Z_t(U)$ contains a forbidden induced subgraph that is a negative odd cycle with a hanging edge, or a positive even cycle with a hanging edge, or $K_{1,4}$ with an edge attached at one of its pendant vertices. The critical cases are considered by inspection, and the lemma follows. □

Next we determine the good vertices for Z_t.

Lemma 3.4. *With the notation above, a vertex u and a corresponding set U are good for Z_t as a star complement for ± 2 if and only if one of the following holds*:

(*i*) (*a*) $U = \{1, 2, t\}$ *with* $\sigma(u1) = -\sigma(u2)$, (*b*) $U = \{1, 2, 4\}$ *with* $\sigma(u1) = \sigma(u2) = -\sigma(u4)$,

(*ii*) $U = \{i, i+2\}$ *with* $3 \leq i \leq t-2$ *and* $\sigma(ui) = -\sigma(u(i+2))$,

(*iii*) $U = \{t-1\}$,

(*iv*) $t = 8$, *and either* U *is one of* $\{i, 4, 6\}, \{i, 4, 8\}, \{i, 6, 8\}, \{i, 4\}, \{i, 6\}, \{i, 8\}$, *where* $i \in \{1, 2\}$ *and the edges from* u *alternate in sign, or* $U = \{i\}$ $(i \in \{1, 2\})$.

Proof. We set $\mathbf{b}_u = \mathbf{b} = (b_1, b_2, \dots, b_t)^\intercal$. Evidently $Z_t(U)$ is good for 2 if and only if $\langle \mathbf{b}, \mathbf{b}\rangle = 2$, where the bilinear form is determined by the matrix (1).

Suppose first that $|U| = 3$, and let u be adjacent to the vertices 1, 2, and i. We have

$$\begin{aligned}4\langle \mathbf{b}, \mathbf{b}\rangle &= 4\mathbf{b}^\intercal(2I - C)^{-1}\mathbf{b}\\ &= b_1(b_1 t + b_2(t-2) + b_i 2(t-i+1)) + b_2(b_1(t-2) + b_2 r + b_i 2(t-i+1))\\ &\quad + b_i(b_1 2(t-i+1) + b_2 2(t-i+1) + b_i 4(t-i+1)).\end{aligned}$$

For $i \neq 4$, we have $b_1 = -b_2$ (otherwise, $Z_t(U)$ contains the positive quadrangle with a hanging edge), and so without loss of generality we may assume that $b_1 = 1$, $b_2 = -1$. This leads to $4\langle \mathbf{b}, \mathbf{b}\rangle = 4(t-i+2)$, and so $\langle \mathbf{b}, \mathbf{b}\rangle = 2$ if and only if $i = t$. For $i = 4$, we obtain $\langle \mathbf{b}, \mathbf{b}\rangle = 2$ with $\sigma(u1) = \sigma(u2) = -\sigma(u4)$. Since in both situations the spectrum of $Z_t(U)$ is symmetric, we also have that U is good for -2, and (i) is completed.

Now suppose that u is adjacent to $1, i$, and j $(3 \leq i < j \leq t)$. In a very similar way, we find that $4\langle \mathbf{b}, \mathbf{b}\rangle = t$ with $\sigma(u1) = -\sigma(ui) = \sigma(uj)$. Hence $t = 8$, with the solutions (iv) computed directly.

Finally, if u is adjacent to i, j, k $(3 \leq i < j < k \leq t)$, then we have $\langle \mathbf{b}, \mathbf{b}\rangle = t - i + j - k + 1$ with $\sigma(ui) = -\sigma(uj) = \sigma(uk)$. The equality $t - i + j - k = 1$ holds only for $k = t$ and $i = j - 1$, but in this case $Z_t(U)$ contains a negative triangle with a hanging edge, and this implies $\lambda(Z_t(U)) < -2$.

Suppose now that $|U| = 2$. Then $U \neq \{1, 2\}$ since in this case either $\Lambda(Z_t) > 2$ or $Z_t(U)$ is a signed line graph of an unbalanced unicyclic simply signed graph with $\lambda(Z_t(U)) > -2$. If u is adjacent to 1 and i $(3 \leq 1 \leq t)$, we find that $4\langle \mathbf{b}, \mathbf{b}\rangle = t$, and as before $t = 8$ along with the given solutions. If u is adjacent to i and j $(3 \leq i < j \leq t)$, we have $\langle \mathbf{b}, \mathbf{b}\rangle = j - i$, and thus $Z_t(U)$ is good for 2 if and only if $j - i = 2$. Simultaneously, $Z_t(U)$ is good for -2 since its spectrum is symmetric. This completes (ii).

For $|U| = 1$ we find easily that $Z_t(U)$ is good for ± 2 if and only if $U = \{1\}$ or $U = \{2\}$, in both cases along with $t = 8$, and $U = \{t-1\}$ (for any t). In this way, we have completed (iv) and (iii). ☐

For the determination of maximal extensions we need the following simple lemma. For a (not necessarily good) vertex u we denote by u^* the vertex obtained by taking a copy of u (with the same neighbourhood in $\dot{H}$) and switching the sign of all edges incident with u^*.

Lemma 3.5. *A vertex u is good for ν if and only if u^* is good for ν. If u is good, then u, u^* are compatible if and only if $\nu \in \{0, 1, -1\}$.*

Proof. With the notation of Theorem 2.1, the first part of this lemma follows from $\langle \mathbf{b}_u, \mathbf{b}_u\rangle = \langle -\mathbf{b}_u, -\mathbf{b}_u\rangle = \langle \mathbf{b}_{u^*}, \mathbf{b}_{u^*}\rangle$. For the second part, if u is good then we have $\nu = \langle \mathbf{b}_u, \mathbf{b}_u\rangle = -\langle \mathbf{b}_u, \mathbf{b}_{u^*}\rangle$, which means that u, u^* are compatible if and only if $\nu \in \{0, 1, -1\}$. ☐

Now we proceed with maximal extensions.

Proposition 3.6. *For $t \neq 8$, $\mathcal{L}(\ddot{C}_t)$ is the unique maximal extension of Z_t* for ± 2.

Proof. We first consider compatibility between the vertices of Lemma 3.4(i)–(iii). It is convenient to compute the vectors $\mathbf{b}_u^\intercal M$ for M defined by (1). With the enumeration of this lemma we find that

(i)

(a) $\mathbf{b}_u^\intercal M = \begin{cases} 4\sigma(ut)(1, 0, 1, 1, \ldots, 1) & \text{for } \sigma(u1) = 1, \\ 4\sigma(ut)(0, 1, 1, 1, \ldots, 1) & \text{for } \sigma(u1) = -1; \end{cases}$

(b) $\mathbf{b}_u^\intercal M = 4\sigma(u1)(1, 1, 1, 0, 0, \ldots, 0)$;

(ii) $\mathbf{b}_u^\intercal M = 4\sigma(ui)(\underbrace{1, 1, 2, 2, \ldots, 2}_{i}, 1, 0, 0, \ldots, 0)$;

(iii) $\mathbf{b}_u^\intercal M = 4\sigma(u(t-1))(1, 1, 2, 2, \ldots, 2, 1)$.

Observe now that a vertex u is never compatible with the vertex u^* defined upon Lemma 3.5. Compatibility between the remaining vertices is given in the scheme below. For example, if u, v are of type (ii), then

$$4\langle \mathbf{b}_u, \mathbf{b}_v\rangle = \mathbf{b}_u^\intercal M \mathbf{b}_v = 4\sigma(ui)(\underbrace{1, 1, 2, 2, \ldots, 2}_{i}, 1, 0, 0, \ldots, 0)\sigma(vj)(\underbrace{0, 0, \ldots, 0}_{j-1}, 1, 0, -1, 0, 0, \ldots, 0)^\intercal,$$

which gives the assertion.

Type of u	Type of v	Compatibility
(i.a)	(i.a)	$u \nsim v$
(i.a)	(i.b)	$u \nsim v$
(i.a)	(ii)	$u \nsim v$
(i.a)	(iii)	$u \sim v$ with $\sigma(uv) = -\sigma(ut)\sigma(v(t-1))$
(i.b)	(ii)	$u \nsim v$, unless $V = \{3, 5\}$ when $u \sim v$ with $\sigma(uv) = \sigma(u4)\sigma(v3)$
(i.b)	(iii)	$u \nsim v$, unless $t = 4$ when $u \sim v$ with $\sigma(uv) = \sigma(u4)\sigma(v3)$
(ii)	(ii)	$u \nsim v$, unless $U = (i, i+2)$, $V = (i+1, i+3)$ when $u \sim v$ with $\sigma(uv) = -\sigma(ui)\sigma(v(i+1))$
(ii)	(iii)	$u \nsim v$, unless $U = (t-2, t)$ when $u \sim v$ with $\sigma(uv) = \sigma(ut)\sigma(v(t-1))$

It follows that good vertices for Z_t can be partitioned into two halves in such a way that for every good vertex u, its non-compatible pair u^* belongs to different half. In addition, by the scheme above all vertices in one half are mutually compatible and comprise a maximal extension, while the vertices of the other half lead to a switching equivalent extension. In other words, there is a unique maximal extension of Z_t (for $t \neq 8$). We construct it on the basis of the vertex compatibility, and obtain $\mathcal{L}(\ddot{C}_t)$. ☐

In the same way, we find that the maximal extension of Z_8 is the second signed graph of Figure 3. For the remaining candidates for star complements (those of (b), (d), and (e)) one can choose between the

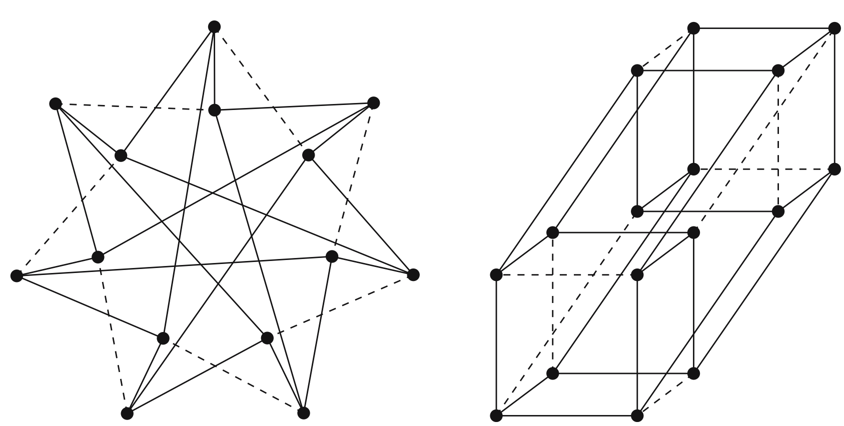

Figure 3: Exceptional signed graphs with spectrum $[2^7, (-2)^7]$ and $[2^8, (-2)^8]$, respectively.

foregoing methods. (Observe that all of them are signed line graphs, and so the first method is applicable.) The first method is restricted to $t \geq 9$, and the resulting maximal extension is unchanged. For $t \leq 8$ and the remaining potential star complements (those of Figure 1 and their connected induced subgraphs) we can choose between a direct computation and brute force, i.e. a computer search which can be performed very quickly since the signed graphs under consideration are of small order. In any case, we obtain just two solutions that deviate from $\mathcal{L}(\ddot{C}_t)$, namely the signed graphs illustrated in Figure 3; one can easily deduce that both are exceptional.

4 Signed graphs with spectrum $[2^t, (-2)^t]$

Now we are ready to give an alternative proof of the result in [1–3].

Theorem 4.1. *A connected signed graph has spectrum* $[2^t, (-2)^t]$ *if and only if it is one of the signed graphs illustrated in* Figures 2 *and* 3.

Proof. With a slight modification in the proof of [8, Theorem 5.1.6] we get that every signed graph has a connected star complement for each of its eigenvalues. Therefore, the result follows unless there is a connected signed graph, say $\dot{G}$, with given spectrum for which there is no common star complement for -2 and 2. In this case, $\dot{G}$ has $2t$ vertices and contains a connected star complement, say $\dot{H}$, for -2, but not for 2, which implies $\Lambda(\dot{H}) = 2$.

If $\dot{G}$ is a signed line graph, then by Lemma 2.2 $\dot{H}$ is the signed line graph of either a tree or an unbalanced unicyclic simply signed graph.

In the former case, we find that $\dot{H}$ contains a positive triangle with a hanging edge (along with a contradiction $\Lambda(\dot{H}) > 2$) unless $\dot{H}$ is a complete unsigned graph, but in this case we have $t \leq 3$, which is easily resolved (a triangle in the role of $\dot{H}$ leads to $\dot{G} \cong \mathcal{L}(\ddot{C}_3)$).

In the latter case, we have the same situation (a positive triangle with a hanging edge) unless $\dot{H}$ is the unbalanced quadrangle with at most two hanging paths (Figure 1) or $\dot{H}$ is a positive cycle. In the first case, contrary to the assumption, we have $\Lambda(\dot{H}) < 2$. In the second case, by considering good vertices we find that $\dot{H}$ can be transformed into a star complement common to both eigenvalues.

The situation in which $\dot{G}$ is exceptional is resolved by inspecting potential star complements with six, seven, and eight vertices. $\square$

We note that exceptional signed graphs with eight vertices of Figure 1 are star complements for the second signed graph of Figure 3, while their induced subgraphs with seven vertices correspond to the first signed graph of the same figure.

We conclude our consideration with the following characterization.

Theorem 4.2. *Every signed graph with spectrum* $[2^t, (-2)^t]$ *is determined by one of the following star complements for* ± 2: Z_t, *or the negative hexagon with a hanging edge, or the negative hexagon with two hanging edges attached at vertices at distance* 3.

Proof. The result follows from Theorem 4.1 and the subsequent discussion. $\square$

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