On star complement technique

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Abstract

The star complement technique is a spectral tool recently developed for constructing some bigger graphs from their smaller parts, called star complements. Here we give a short description of this technique, while more details one can find in [4] and [6].

We consider only simple graphs, that is finite, undirected graphs without loops or multiple edges. If G is such a graph with vertex set $V_G = \{1, 2, \dots, n\}$, the *adjacency matrix* of G is $n \times n$ matrix $A_G = (a_{ij})$, where $a_{ij} = 1$ if there is an edge between the vertices i and j, and 0 otherwise. The *eigenvalues* of G, denoted by

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n,$$

are just the eigenvalues of A_G. Note, the eigenvalues of G are real and do not depend on vertex labelling. Additionally, for connected graphs $\lambda_1 > \lambda_2$ holds. The *characteristic polynomial* of G is the characteristic polynomial of its adjacency matrix, so $P_G(\lambda) = \det(\lambda I - A_G)$. For more details on graph spectra, see [3].

If μ is an eigenvalue of G of multiplicity k, then a *star set* for μ in G is a set X of k vertices taken from G such that μ is not an eigenvalue of $G - X$. The graph $H = G - X$ is then called a *star complement* for μ in G (or a *μ–basic subgraph* of G in [7]). (Star sets and star complements exist for any eigenvalue and any graph; they need not be unique.) The H–neighborhoods of vertices in X can be shown to be non–empty and distinct, provided that $\mu \notin \{-1, 0\}$ (see [5], Chapter 7). If $t = |V_H|$, then $|X| \leq \binom{t}{2}$ (see [1]) and this bound is best possible.

It can be proved that if Y is a proper subset of X then $X - Y$ is a star set for μ in $G - Y$, and therefore H is a star complement for μ in $G - Y$. If G has star complement H for μ, and G is not a proper induced subgraph of some other graph with star complement H for μ, then G is a *maximal graph* with star complement H for μ, or it is an *H–maximal graph* for μ. By the above remarks, there are only finitely many such maximal graphs, provided $\mu \notin \{-1, 0\}$. In general, there will be only several maximal graphs, possibly of different orders, but sometimes there is a unique maximal graph (if so, this graph is characterized by its star complement for μ).

We now mention some results from the literature (they are taken from [4], [5] and [6]).

The following result is known as the Reconstruction Theorem (see, for example, [5], Theorems 7.4.1 and 7.4.4).

Theorem 1 *Let G be a graph with adjacency matrix*

$$\begin{pmatrix} A_X & B^T \\ B & C \end{pmatrix},$$

where A_X is the adjacency matrix of the subgraph induced by the vertex set X. Then X is a star set for μ if and only if μ is not an eigenvalue of C and $\mu I - A_X = B^T(\mu I - C)^{-1}B$.

From the above, we see that if μ, C and B are fixed then A_X is uniquely determined. In other words, given the eigenvalue μ, a star complement H for μ, and the H–neighborhoods of the vertices in the star set X, the graph G is uniquely determined. In the light of these facts, we may next ask to what extent G is determined only by H and μ. Having in mind the observation above, it is sufficient to consider graphs G which are H–maximal for μ.

Following [2], [8] and [9], we list some notation and terminology. Given a graph H, a subset U of $V(H)$ and a vertex u not in $V(H)$, denote by $H(U)$ the graph obtained from H by joining u to all vertices of U. We will say that u (U, $H(U)$) is a *good vertex* (resp. *good set*, *good extension*) for μ and H, if μ is an eigenvalue of $H(U)$ but is not an eigenvalue of H. By Theorem 1, a vertex u and a subset U are good if and only if $\mathbf{b}_u^T(\mu I - C)^{-1}\mathbf{b}_u = \mu$, where $\mathbf{b}_u$ is the characteristic vector of U (with respect to $V(H)$) and C is the adjacency matrix of H. Assume now that U_1 and U_2 are not necessarily good sets corresponding to vertices u_1 and u_2, respectively. Let $H(U_1, U_2; 0)$ and $H(U_1, U_2; 1)$ be the graphs obtained by adding to H both vertices, u_1 and u_2, so that they are non–adjacent in the former graph, while adjacent in the latter graph. We say that u_1 and u_2 are *good partners* and that U_1 and U_2 are *compatible sets* if μ is an eigenvalue of multiplicity two either in $H(U_1, U_2; 0)$ or in $H(U_1, U_2; 1)$. (Note, if $\mu \notin \{-1, 0\}$, any good set is non–empty, any two of them if corresponding to compatible sets are distinct; see [5], Proposition 7.6.2.) By Theorem 1, two vertices u_1 and u_2 are good partners (or two sets U_1 and U_2 are compatible) if and only if $\mathbf{b}_{u_1}^T(\mu I - C)^{-1}\mathbf{b}_{u_2} \in \{-1, 0\}$, where $\mathbf{b}_{u_1}$ and $\mathbf{b}_{u_2}$ are defined as above. In addition, it follows (again by Theorem 1) that any vertex set X in which all vertices are good, both individually and in pairs, gives rise to a *good extension*, say G, in which X can be viewed as a star set for μ, while H as the corresponding star complement.

The above considerations shows us how we can introduce a technique, also called a *star complement technique*, for finding (or constructing) graphs with certain spectral properties. In this context the graphs we are interested in should have some prescribed eigenvalue usually of a very large multiplicity. If G is a graph in which μ is an eigenvalue of multiplicity $k > 1$, then G is a good (k–vertex) extension of some of its star complements, say H (in particular, G is H–maximal for μ). The *star complement* technique consists of the following: In order to find H–maximal graphs for μ ($\neq -1, 0$), we form an *extendability graph* whose vertices are good vertices for μ and H, and add an edge between two good vertices whenever they are good partners. Now it is easy to see that the search for maximal extensions is reduced to the search for maximal cliques in the extendability graph (see, for example, [4] and [6]). Of course, among H–maximal graphs some of them can be mutually isomorphic. So, we determine how many different isomorphism classes they belong to.

References

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